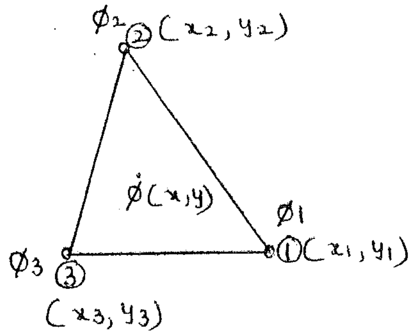
UNIT – IV :-

TWO – DIMENSIONAL PROBLEM :-

Finite Element Formulation :-

ⓐ Cartesian co-ordinates :-



ϕ_1 = value of ϕ at node 1.

ϕ_2 = value of ϕ at node 2.

ϕ_3 = value of ϕ at node 3.

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y.$$

$\phi = \phi_1$ @ $x = x_1, y = y_1$ $\qquad \phi_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$

$\phi = \phi_2$ @ $x = x_2, y = y_2$ $\qquad \phi_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$

$\phi = \phi_3$ @ $x = x_3, y = y_3$ $\qquad \phi_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3.$

By solving the above three Equations, we get

$$\alpha_1 = \frac{1}{2A}(a_1\phi_1 + a_2\phi_2 + a_3\phi_3)$$

$$\alpha_2 = \frac{1}{2A}(b_1\phi_1 + b_2\phi_2 + b_3\phi_3)$$

$$\alpha_3 = \frac{1}{2A}(c_1\phi_1 + c_2\phi_2 + c_3\phi_3)$$

* where $A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

where

* $a_1 = (x_2 y_3 - x_3 y_2)$ $\qquad$ * $b_1 = y_2 - y_3$ $\qquad$ * $c_1 = x_3 - x_2$
* $a_2 = (x_3 y_1 - x_1 y_3)$ $\qquad$ * $b_2 = y_3 - y_1$ $\qquad$ * $c_2 = x_1 - x_3$
* $a_3 = (x_1 y_2 - x_2 y_1)$ $\qquad$ * $b_3 = y_1 - y_2$ $\qquad$ * $c_3 = x_2 - x_1.$

'ϕ' can be written as

$$\boxed{\phi = N_1\phi_1 + N_2\phi_2 + N_3\phi_3}$$

ii) By using interpolation formulation N_1, N_2 and N_3 are called as Shape function.

where $* N_1 = \frac{1}{2A} (a_1 + b_1 x + c_1 y)$

$$* N_2 = \frac{1}{2A} (a_2 + b_2 x + c_2 y)$$

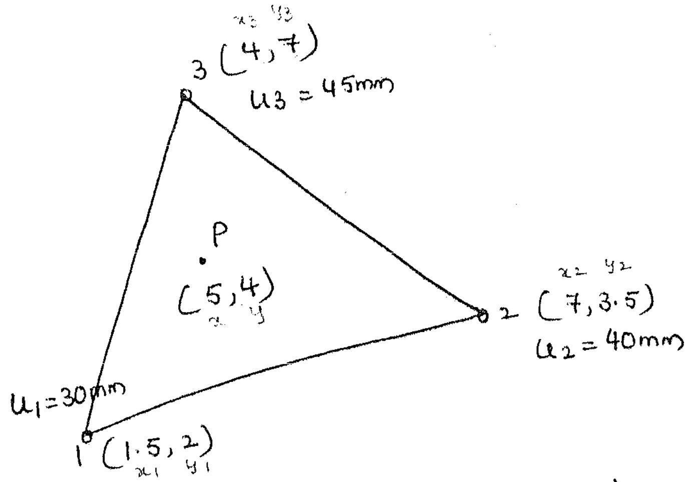
$$* N_3 = \frac{1}{2A} (a_3 + b_3 x + c_3 y)$$

where N_1, N_2 and N_3 are called shape functions. ie, once coordinates are known, shape functions can be calculated. $\{ N_1 + N_2 + N_3 = 1 \}$

* characteristics of shape functions :-

1. Value of shape function at the node for which it belongs is equal to 1.
2. Value of shape function at other nodes = 0.
3. At other points, sum of shape functions is equal to 1.

Prob:1



Find displacement at P (5, 4).

Sol: Given that $u_1 = 30mm$

$u_2 = 40mm$

$u_3 = 45 mm$

$$\text{Area, } A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1.5 & 2 \\ 1 & 7 & 3.5 \\ 1 & 4 & 7 \end{vmatrix}$$

$$A = \frac{1}{2}\left[1(7\times7 - 4\times3.5) - 1.5(1\times7 - 1\times3.5) + 2(1\times4 - 1\times7)\right]$$

$$A = \frac{1}{2}\left[(49-14) - 1.5(7-3.5) + 2(4-7)\right]$$

$$A = \frac{1}{2}\left[35 - 1.5\times3.5 + 2\times(-3)\right]$$

$$A = \frac{1}{2}\left[35 - 5.25 - 6\right]$$

$$\boxed{A = 11.875}$$

And also

$$a_1 = x_2 y_3 - x_3 y_2 = 7\times7 - 4\times3.5 = 35$$

$$a_2 = x_3 y_1 - x_1 y_3 = 4\times2 - 1.5\times7 = -2.5$$

$$a_3 = x_1 y_2 - x_2 y_1 = 1.5\times3.5 - 7\times2 = -8.75$$

$$b_1 = y_2 - y_3 = 3.5 - 7 = -3.5$$

$$b_2 = y_3 - y_1 = 7 - 2 = 5$$

$$b_3 = y_1 - y_2 = 2 - 3.5 = -1.5$$

$$c_1 = x_3 - x_2 = 4 - 7 = -3.0$$

$$c_2 = x_1 - x_3 = 1.5 - 4 = -2.5$$

$$c_3 = x_2 - x_1 = 7.0 - 1.5 = +5.5$$

Shape functions at point $p(x,y) = P(5,4)$

$$N_1 = \frac{1}{2A}(a_1 + b_1 x + c_1 y)$$

$$= \frac{1}{2\times11.875}\left(35 + (-3.5)(5) + (-3)(4)\right)$$

$$\boxed{N_1 = 0.2315}$$

$$N_2 = \frac{1}{2A}(a_2 + b_2 x + c_2 y)$$

$$= \frac{1}{2\times 11.875}(-2.5 + 5\times 5 + (-2.5)4)$$

$$\boxed{N_2 = 0.5263}$$

$$N_3 = \frac{1}{2A}(a_3 + b_3 x + c_3 y)$$

$$= \frac{1}{2\times 11.875}(-8.75 + (-1.5)(5) + (5.5)(4))$$

$$\boxed{N_3 = 0.2421}$$

Check: $N_1 + N_2 + N_3 = 0.2315 + 0.5263 + 0.2421 = 0.9999 \approx 1.0$

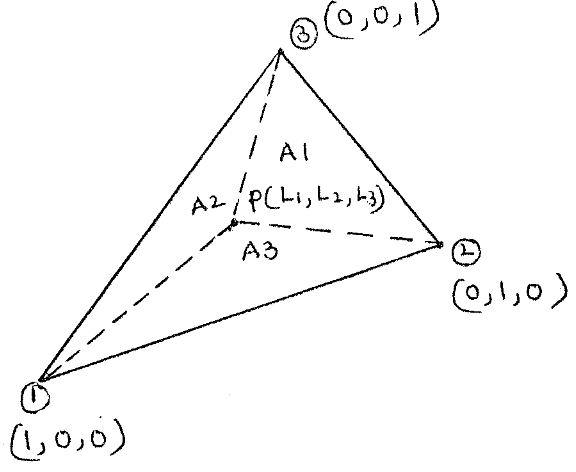
Then Displacement at $P(5,4)$,

$$U = N_1 U_1 + N_2 U_2 + N_3 U_3$$

$$= 0.2315 \times 30 + 0.5263 \times 40 + 0.2421 \times 45$$

$$\boxed{U = 38.89 \text{ mm}}$$

(b) Natural co-ordinate Method :-



A_1 = Area opp to node 1.

A_2 = Area opp to node 2.

A_3 = Area opp to node 3.

Here $L_1 = \frac{A_1}{A}$ $\quad N_1 = L_1$

$L_2 = \frac{A_2}{A}$ $\quad N_2 = L_2$

$L_3 = \frac{A_3}{A}$ $\quad N_3 = L_3$

$\therefore \phi = N_1\phi_1 + N_2\phi_2 + N_3\phi_3.$

$\therefore L_1, L_2, L_3$ for node ① are

$L_1 = A/A = 1$

$L_2 = 0/A = 0$

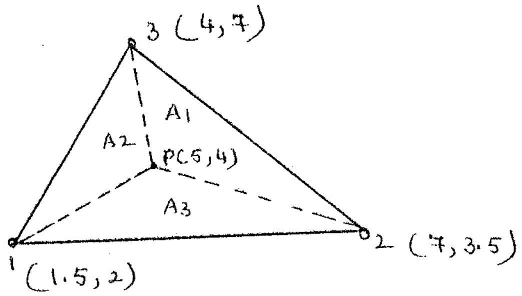
$L_3 = 0/A = 0$

And area 'A' is given by

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad A_2 = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{vmatrix}, \quad A_3 = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}$$

prob: Using Natural co-ordinate system, find displacement at P(5, 4).



Sol: Given that $x_1 = 1.5$, $y_1 = 2$

$x_2 = 7$, $y_2 = 3.5$

$x_3 = 4$; $y_3 = 7$

$$A = \frac{1}{2}\begin{vmatrix} 1 & 1.5 & 2 \\ 1 & 7 & 3.5 \\ 1 & 4 & 7 \end{vmatrix} = \frac{1}{2}\left[(49-14) - 1.5(7-3.5) + 2(4-7)\right]$$

$$= \frac{1}{2}\left[35 - 5.25 + 2(-3)\right]$$

$$\boxed{A = 11.875}$$

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & 5 & 4 \\ 1 & 7 & 3.5 \\ 1 & 4 & 7 \end{vmatrix} = \frac{1}{2}\left[(49-14) - 5(3.5) + 4(-3)\right]$$

$$\boxed{A_1 = 2.75}$$

$$A_2 = \frac{1}{2}\begin{vmatrix} 1 & 1.5 & 2 \\ 1 & 5 & 4 \\ 1 & 4 & 7 \end{vmatrix} = \frac{1}{2}\left[(35-16) - 1.5(3) + 2(-1)\right]$$

$$\boxed{A_2 = 6.25}$$

$$A_3 = \frac{1}{2}\begin{vmatrix} 1 & 1.5 & 2 \\ 1 & 7 & 3.5 \\ 1 & 5 & 4 \end{vmatrix} = \frac{1}{2}\left[(28-17.5) - 0.5 \times 1.5 + 2(-2)\right]$$

$$= \frac{1}{2} \times 5.75$$

$$\therefore \boxed{A_3 = 2.875}$$

Then $L_1 = \dfrac{A_1}{A} = \dfrac{2.75}{11.875} = 0.231$

$L_2 = \dfrac{A_2}{A} = \dfrac{6.25}{11.875} = 0.5263$

$L_3 = \dfrac{A_3}{A} = \dfrac{2.875}{11.875} = 0.242$

$\therefore N_1 = L_1 = 0.231$

$N_2 = L_2 = 0.5263$

$N_3 = L_3 = 0.242$

Check: $N_1 + N_2 + N_3 = 0.231 + 0.5263 + 0.242 = 0.9993 \simeq 1.$

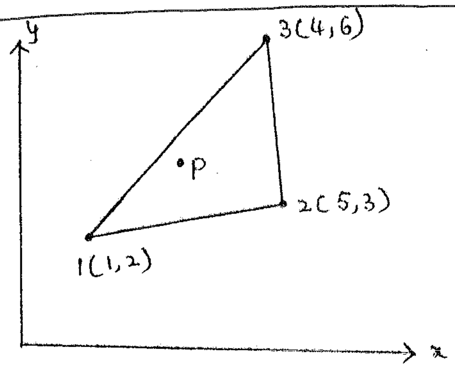
Then displacement at $P(5,4)$.

$$U = N_1 U_1 + N_2 U_2 + N_3 U_3$$

$$= 0.231 \times 30 + 0.5263 \times 40 + 0.242 \times 45$$

$$\boxed{U = 38.872 \text{mm.}}$$

Prob: The nodal co-ordinates of the triangular element are shown in fig. At the interior point P, the x-coordinate is 3.3, and $N_1 = 0.3$. Determine N_2, N_3, and the y-coordinate at point P.



Sol: Given Data:

$N_1 = 0.3$

$x = 3.3$

$x_1 = 1, \; y_1 = 2$

$x_2 = 5, \; y_2 = 3$

$x_3 = 4, \; y_3 = 6.$

By using Natural co-ordinate Method.

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & 1 & 2 \\ 1 & 5 & 3 \\ 1 & 4 & 6 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(30-12) - 1(6-3) + 2(4-5)\right]$$

$$= \frac{1}{2}\left[18 - 3 - 2\right]$$

$$\boxed{A = 6.5}$$

$$A_1 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 3.3 & y \\ 1 & 5 & 3 \\ 1 & 4 & 6 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(30-12) - 3.3(6-3) + y(4-5)\right]$$

$$= \frac{1}{2}\left[18 - 3.3\times 3 - y\right]$$

$$= \frac{1}{2}\left[8.1 - y\right]$$

$$A_1 = 4.05 - 0.5y$$

But $\quad L_1 = N_1 = \dfrac{A_1}{A}$

$$0.3 = \frac{(4.05 - 0.5y)}{6.5}$$

$$4.05 - 0.5y = 0.3 \times 6.5$$

$$0.5y = 4.05 - 0.3 \times 6.5$$

$$y = \frac{2.1}{0.5} \Rightarrow * \boxed{y = 4.2}$$

$$A_2 = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 2 \\ 1 & 3.3 & 4.2 \\ 1 & 4 & 6 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(3.3 \times 6 - 4.2 \times 4) - 1(1 \times 6 - 1 \times 4.2) + 2(1 \times 4 - 3.3 \times 1)\right]$$

$$= \frac{1}{2}\left[3 - 1.8 + 1.4\right]$$

$$\boxed{A_2 = 1.30}$$

$$A_3 = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 2 \\ 1 & 5 & 3 \\ 1 & 3.3 & 4.2 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(5 \times 4.2 - 3.3 \times 3) - 1(1 \times 4.2 - 1 \times 3) + 2(1 \times 3.3 - 1 \times 5)\right]$$

$$= \frac{1}{2}\left[11.1 - 1.2 - 3.4\right]$$

$$\boxed{A_3 = 3.25}$$

Note: $A_1 + A_2 + A_3 = 1.95 + 1.3 + 3.25 = 6.5 = A$ (Check)

Then the required shape mode function,

$$L_2 = N_2 = \frac{A_2}{A}$$

$$= \frac{1.3}{6.5}$$

$$\boxed{N_2 = 0.2}$$

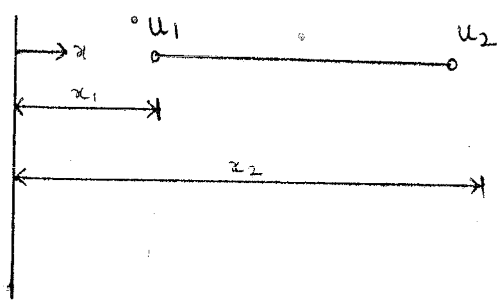
$$L_3 = N_3 = \frac{A_3}{A}$$

$$= \frac{3.25}{6.50}$$

$$\boxed{N_3 = 0.5}$$

Check: $N_1 + N_2 + N_3 = 0.3 + 0.2 + 0.5 = 1.0$ ✓

FINITE ELEMENT FORMULATION :-



coordinate of node ① : x_1

coordinate of node ② : x_2

Let us assume an interpolation / displacement model as

$$u = \alpha_1 + \alpha_2 x$$

at $x = x_1$, $u = u_1$

at $x = x_2$, $u = u_2$

$$\therefore u_1 = \alpha_1 + \alpha_2 x_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2$$

Solving for the two unknowns α_1 & α_2.

$$\alpha_1 = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1}$$

$$\alpha_2 = \frac{u_2 - u_1}{x_2 - x_1}$$

$$\therefore u = \left[\frac{u_1 x_2 - u_2 x_1}{x_2 - x_1}\right] + \left[\frac{u_2 - u_1}{x_2 - x_1}\right] x$$

$$u = \frac{x_2 - x}{x_2 - x_1} \cdot u_1 + \frac{x - x_1}{x_2 - x_1} \cdot u_2$$

$$u = N_1 u_1 + N_2 u_2$$

where $N_1 = \dfrac{x_2 - x}{x_2 - x_1}$ & $N_2 = \dfrac{x - x_1}{x_2 - x_1}$ } which are called shape functions.

$\therefore u = N_1 u_1 + N_2 u_2$ is a better way of assuming an interpolation model rather than as $u = \alpha_1 + \alpha_2 x$.

In general,

$$\bar{u} = \sum_{i=1}^{n} N_i u_i$$

where n = no. of nodes.

$$\bar{u} = N_1 U_1 + N_2 U_2 + N_3 U_3 + \ldots$$

$$\bar{u} = [N][U]$$

where $N = [N_1 \quad N_2 \quad \ldots]$

$$U = \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \end{Bmatrix}$$

Total potential Energy,

$$\Pi = \frac{1}{2}\int \sigma^T \epsilon \, dv - \int u^T f \, dv - \int u^T T \, ds - \sum_{i=1}^{K} u_i p_i \longrightarrow (1)$$

Internal strain Energy; Due to body forces; Due to Surface Traction forces; Due to point Loads.

$$\bar{\sigma} = \bar{D}\,\bar{\epsilon}$$

$$\sigma^T = \epsilon^T D^T = \epsilon^T . D$$ is a Symmetric matrix.

Equation (1) is for 2 Dimensional object,

$$\bar{u} = \bar{N}\,\bar{U}$$

For 2D Elements :

$$\begin{bmatrix} u^{(e)} \\ v^{(e)} \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i^{(e)} & 0 \\ 0 & N_i^{(e)} \end{bmatrix} \begin{bmatrix} u_i^{(e)} \\ v_i^{(e)} \end{bmatrix}$$

'e' stands for Element.

<u>Ex:</u> For a 2D problem,

2 displacements 'u' & 'v'.

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$\bar{u} = \bar{N}\,\bar{U}$$

$$\{\because (AB)^T = B^T A^T\}$$

$$u^T = U^T N^T = U^T N$$

consider the Second part ii,

the load vector of total Potential Energy expression,

which,

$$= -\left[\int u^T \bar{f}\, dv + \int u^T T\, dA + \sum_{i=1}^{K} u_i p_i\right]^\circ$$

$$= \left[\int U^T N^T f\, dv + \int U^T N^T T\, dA + \sum_{i=1}^{K} u_i p_i\right]$$

using the notation:

$$\bar{\epsilon} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} u_{1x} \\ v_{1y} \\ u_{1y} + v_{1x} \end{bmatrix}$$

$$u_{1x} = \frac{\partial N_i}{\partial x} U$$

$$= N_{ix} \cdot U$$

$$\begin{bmatrix} u_{1x} \\ v_{1y} \\ u_{1y} + v_{1x} \end{bmatrix} = \sum_{i=1}^{n} \underbrace{\begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix}}_{\text{This is called B-matrix or strain - Displacement matrix.}} \underbrace{\begin{bmatrix} u_i \\ v_i \\ \end{bmatrix}}_{\bar{U}}$$

$$\bar{\epsilon} = \bar{B}\,\bar{U}$$

$$\epsilon^T = U^T B^T$$

$$\pi = \frac{1}{2}\int U^T B^T D B U \cdot dv - \int U^T . F$$

For minimum potential Energy, $\frac{\partial \pi}{\partial U} = 0$ gives

$$\underbrace{\int B^T D B \, dv}\cdot U - F = 0$$

This is nothing but 'K' matrix :

$$\therefore \quad \bar{K}\,\bar{U} = \bar{F}$$

$\therefore$ For any Element,

$$K = \int B^T D B \, dv \quad ; \quad \text{Global } K = \sum_{i=1}^{n} K_i$$

Load vector $= F = \int N^T f \, dv + \int N^T T \, ds + P$

$$\sigma = D \cdot \epsilon$$

$$= \bar{D} \cdot \bar{B} \cdot \bar{U} \qquad \text{because } \epsilon = \bar{B} \cdot \bar{U}$$

For one dimensional problem,

$$dv = A\,dx.$$

For two dimensional problem,

$$\iint dv = t \cdot dx \cdot dy.$$

For three dimensional problem,

$$\iiint dv = dx\,dy\,dz.$$

$$u = N_1 U_1 + N_2 U_2$$

$$\bar{u} = \bar{N}\,\bar{U}$$

$$= \bar{N}\,\bar{q}$$ ($\bar{u}, \bar{q}$ is used ~~for~~ $\bar{U}$ only to avoid confusion).

$$\therefore \quad N_1 = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$$

$$q = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Strain :-

$$\epsilon = \bar{B}\,\bar{q}$$

$$N_1 = \frac{x_2 - x}{x_2 - x_1}, \quad N_2 = \frac{x - x_1}{x_2 - x_1}$$

$$\epsilon = \frac{du}{dx}$$ in one - Dimensional Elements.

$$u = \left(\frac{x_2 - x}{x_2 - x_1}\right) u_1 + \left(\frac{x - x_1}{x_2 - x_1}\right) u_2 .$$

$$\frac{du}{dx} = \frac{u_2 - u_1}{x_2 - x_1}$$

$$= \frac{1}{x_2 - x_1}\left[-u_1 + u_2\right]$$

$$= \underbrace{\frac{1}{x_2 - x_1}\begin{bmatrix}-1 & 1\end{bmatrix}}_{\bar{B}} \underbrace{\begin{bmatrix}u_1 \\ u_2\end{bmatrix}}_{\bar{q}}$$

For one dimensional element, $D = E$

$$K = \int B^T D B \cdot dv$$

$$= \int B^T E B \cdot A\,dx$$

For 1-D element, E = constant

A = constant.

$$\therefore K = EA \int_{x_1}^{x_2} B^T B\,dx$$

$$B = \frac{1}{(x_2 - x_1)}\begin{bmatrix}-1 & 1\end{bmatrix}$$

$$B^T = \frac{1}{x_2 - x_1}\begin{bmatrix}-1 \\ 1\end{bmatrix}$$

$$B^T B = \frac{1}{x_2 - x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \&$$

$$K = \frac{AE}{x_2 - x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Similarly

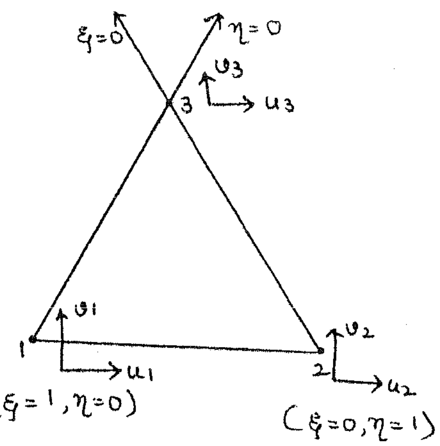
$$\int N^T f \, dv = \frac{Af}{2} l \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\int N^T T \, dA = \frac{Tl}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Local coordinate system (ξ, η) :-
$\downarrow$
(Xi, eta)

ISoparametric Elements :- TWO DIMENSIONAL ELEMENTS

ISoparametric functions (N_1, N_2, N_3).



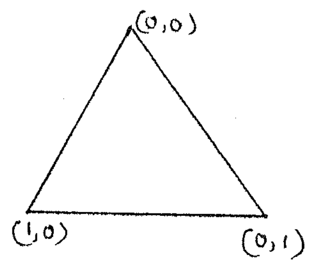
In ISoparametric representation :

$$N_1 = \xi$$

$$N_2 = \eta$$

$$N_3 = 1 - \xi - \eta.$$

∴ In ISoparametric representation, the coordinates of nodes are



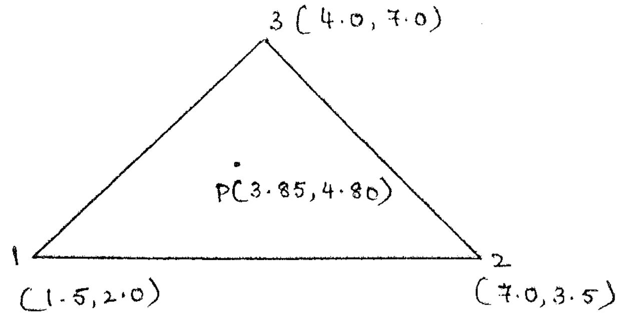
$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3.$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

Prob: Find the values of N_1, N_2 & N_3 at point 'p'.



Sol: For a 3 noded triangular Element.

$$N_1 = \xi$$

$$N_2 = \eta$$

$$N_3 = 1 - \xi - \eta$$

we have $x = N_1 x_1 + N_2 x_2 + N_3 x_3$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$\therefore \; 3.85 = \xi \times 1.5 + \eta \times 7.0 + (1-\xi-\eta) \times 4.0$$

$$= -2.5\xi + 3\eta + 4$$

ie, $2.5\xi - 3\eta = 0.15 \longrightarrow (1)$

Similarly,

$$4.80 = \xi \times 2.0 + \eta \times 3.5 + (1-\xi-\eta)\, 7.0$$

$$= -5\xi - 3.5\eta + 7.0$$

ie, $5\xi + 3.5\eta = 2.20 \longrightarrow (2)$

By solving (1) and (2) Equations, we get

$$\xi = 0.3$$

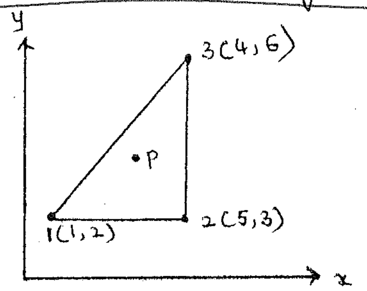
$$\eta = 0.2$$

$$\therefore \; \boxed{\begin{aligned} N_1 &= \xi = 0.3 \\ N_2 &= \eta = 0.2 \\ N_3 &= 1-\xi-\eta = 0.5 \end{aligned}}$$

$u = 0.3u_1 + 0.2u_2 + 0.5u_3$

$v = 0.3v_1 + 0.2v_2 + 0.5v_3$

Prob: The nodal coordinates of the triangular element are shown in fig. At the interior point p, the x-coordinate is 3.3 and $N_1 = 0.3$. Determine N_2, N_3 and the y-coordinate at point p.



Sol: Given Data:

$$N_1 = 0.3 = \xi$$

x-coordinate, $x = 3.3$

$x_1 = 1, \quad y_1 = 2$

$x_2 = 5, \quad y_2 = 3$

$x_3 = 4, \quad y_3 = 6.$

By using Isoparametric Element Method:

$$* \; N_1 = \xi$$

$$* \; N_2 = \eta$$

$$* \; N_3 = 1 - \xi - \eta$$

we have $x = N_1 x_1 + N_2 x_2 + N_3 x_3$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

ie, $x = \xi x_1 + \eta x_2 + (1-\xi-\eta) x_3$

$$y = \xi y_1 + \eta y_2 + (1-\xi-\eta) y_3$$

$$\Rightarrow \quad 3.3 = 0.3 \times 1 + \eta(5) + (1 - 0.3 - \eta) 4$$

$$\Rightarrow \quad 3.3 = 0.3 + 5\eta + 0.7 \times 4 - 4\eta$$

$$\Rightarrow \quad 3.3 = \eta + 3.1$$

$$\Rightarrow \quad \boxed{\eta = 0.2}$$

$\therefore$ $\boxed{N_2 = \eta = 0.2}$

Similarly, $y = \xi y_1 + \eta y_2 + (1-\xi-\eta) y_3$

$y = 0.3\times2 + 0.2\times3 + (1-0.3-0.2)\times 6$

$y = 0.6 + 0.6 + 0.5\times 6$

$\boxed{y = 4.2}$

And also, * $N_3 = 1-\xi-\eta$

$= 1 - 0.3 - 0.2$

$\boxed{N_3 = 0.5}$

Hence the required Isoparametric function and y-value are

$$\boxed{\begin{array}{l} N_2 = 0.2 \\ N_3 = 0.5 \quad \& \\ y = 4.2 \end{array}}$$

Evaluate the Strains:-

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$

But
$$\left.\begin{aligned} \frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi} \\ \frac{\partial u}{\partial \eta} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta} \end{aligned}\right\} \text{Partial derivatives of 'u'.}$$

Expressing in Matrix form,

$$\begin{bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

This matrix is called 'Jacobian matrix' or simply 'J' matrix

$$\therefore \quad J = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 \quad ; \quad y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$= \xi x_1 + \eta x_2 + (1-\xi-\eta) x_3 \; ; \quad = \xi y_1 + \eta y_2 + (1-\xi-\eta) y_3$$

$$\frac{\partial x}{\partial \xi} = (x_1 - x_3) \quad ; \quad \frac{\partial y}{\partial \xi} = (y_1 - y_3)$$

$$\frac{\partial x}{\partial \eta} = (x_2 - x_3) \quad ; \quad \frac{\partial y}{\partial \eta} = (y_2 - y_3)$$

$$J = \begin{bmatrix} (x_1 - x_3) & (y_1 - y_3) \\ (x_2 - x_3) & (y_2 - y_3) \end{bmatrix} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

Now $$\begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{bmatrix} , \quad \text{III}^{ly} \begin{bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \partial v / \partial \xi \\ \partial v / \partial \eta \end{bmatrix}$$

J^{-1} is given by :

$$J^{-1} = \frac{1}{\det J} \cdot \cancel{\text{Adj}}\, J$$

$$\boxed{J^{-1} = \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}}$$

$$\therefore \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} y_{23} \cdot \dfrac{\partial u}{\partial \xi} - y_{13} \cdot \dfrac{\partial u}{\partial \eta} \\ -x_{23} \dfrac{\partial u}{\partial \xi} + x_{13} \dfrac{\partial u}{\partial \eta} \end{bmatrix}$$

{∵ Replacing 'u' by the displacement 'v'} for 'v'.

we have,

$$\epsilon = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix}$$

$$= \frac{1}{\det J} \begin{bmatrix} y_{23} \dfrac{\partial u}{\partial \xi} - y_{13} \dfrac{\partial u}{\partial \eta} \\ -x_{23} \dfrac{\partial v}{\partial \xi} + x_{13} \dfrac{\partial v}{\partial \eta} \\ -x_{23} \dfrac{\partial u}{\partial \xi} + x_{13} \dfrac{\partial u}{\partial \eta} + y_{23} \dfrac{\partial v}{\partial \xi} - y_{13} \dfrac{\partial v}{\partial \eta} \end{bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 = \xi u_1 + \eta u_2 + (1-\xi-\eta) u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 = \xi v_1 + \eta v_2 + (1-\xi-\eta) v_3$$

$$\therefore \frac{\partial u}{\partial \xi} = u_1 - u_3, \quad \frac{\partial u}{\partial \eta} = u_2 - u_3$$

$$\frac{\partial v}{\partial \xi} = v_1 - v_3; \quad \frac{\partial v}{\partial \eta} = v_2 - v_3$$

$$\therefore \epsilon = \frac{1}{\det J} \begin{bmatrix} y_{23}(u_1 - u_3) - y_{13}(u_2 - u_3) \\ -x_{23}(v_1 - v_3) + x_{13}(v_2 - v_3) \\ -x_{23}(u_1 - u_3) + x_{13}(u_2 - u_3) + y_{23}(v_1 - v_3) - y_{13}(v_2 - v_3) \end{bmatrix}$$

From the definition of x_{ij} and y_{ij}; we can write $y_{31} = -y_{13}$ and $y_{12} = y_{13} - y_{23}$, and so on. The foregoing equation can be written in the form:

$$\epsilon = \frac{1}{\det J}\begin{bmatrix} y_{23}u_1 + y_{31}u_2 + y_{12}u_3 \\ x_{32}v_1 + x_{13}v_2 + x_{21}v_3 \\ x_{32}u_1 + x_{13}u_2 + x_{21}u_3 + y_{23}v_1 + y_{31}v_2 + y_{12}v_3 \end{bmatrix}$$

This equation can be written in matrix form as,

$$\epsilon = \bar{B}\bar{q}$$

where 'B' is a (3×6) element strain – displacement matrix relating the three strains to the six nodal displacements and is given by

$$B = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$\epsilon = \underbrace{\frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}}_{\bar{B}} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}}_{\bar{q} \text{ (or) } \bar{U}}$$

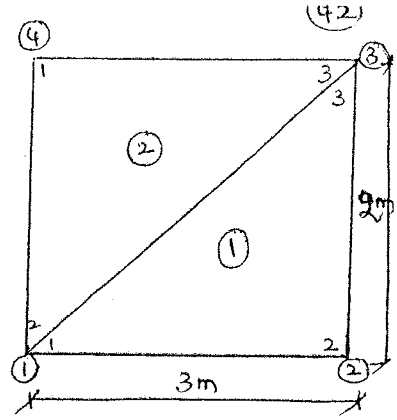
once 'B' matrix is there,

$$\text{Stiffness} = \int B^T D B \cdot dA$$

while working out problems, first do this, as the same can be repeatedly used in calculation of stress etc.

* Stress $= D \cdot \epsilon$
$= \bar{D} \cdot \bar{B} \cdot \bar{q}$.

Prob: Plate divided in to two elements. calculate B^1 and B^2 i.e., 'B' for element ①
'B' for element ②



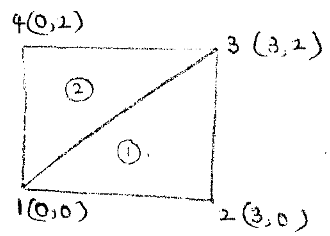
Sol: ⓐ Element connectivity

Element	1	2	3	Local nos
①	1	2	3	Global nos
②	4	1	3	

anti clockwise direction

ⓑ Element ①:

$$\text{Det } J^1 = \begin{vmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{vmatrix}$$



(Local co-ordinates)

$$= \begin{vmatrix} 0-3 & 0-2 \\ 3-3 & 0-2 \end{vmatrix}$$

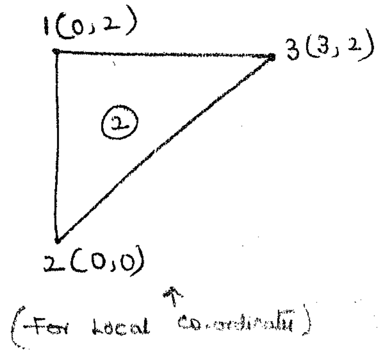
$$= \begin{vmatrix} -3 & -2 \\ 0 & -2 \end{vmatrix} = 6.$$

$$B^1 = \frac{1}{\text{Det } J^1} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} (0-2) & 0 & (2-0) & 0 & (0-0) & 0 \\ 0 & (3-3) & 0 & (0-3) & 0 & (3-0) \\ (3-3) & (0-2) & (0-3) & (2-0) & (3-0) & (0-0) \end{bmatrix}.$$

$$B^1 = \frac{1}{6}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -2 & -3 & 2 & 3 & 0 \end{bmatrix}$$

ⓒ <u>Element ② :-</u>



$$\text{Det } J^2 = \begin{vmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{vmatrix}$$

$$= \begin{vmatrix} 0-3 & 2-2 \\ 0-3 & 0-2 \end{vmatrix} = 6$$

$$B^2 = \frac{1}{\text{Det } J^2}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} (0-2) & 0 & (2-2) & 0 & (2-0) & 0 \\ 0 & (3-0) & 0 & (0-3) & 0 & (0-0) \\ (3-0) & (0-2) & (0-3) & (2-2) & (0-0) & (2-0) \end{bmatrix}$$

$$B^2 = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}_{3\times 6}$$

←

CONSTANT STRAIN TRIANGLE :- (CST):

The B-matrix for a 3 noded triangular element is called constant strain Triangle (CST), because all the terms in the matrix are constant.

Stiffness matrix $= K = \int B^T D B \, dA$ for 1D element

$= \int B^T D B \, dV$ for 2D element

$= t \int B^T D B \, dA$

$\therefore$ For 2D element, $K = t \cdot B^T D B A$.

$\therefore$ Stiffness matrix for a CST $= \boxed{K = t B^T D B A}$

Loads:-

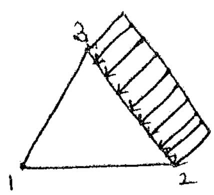
(i) Body forces:-

f_x = Body force in x-direction

f_y = Body force in y-direction.

$$f = \frac{tA}{3}\left[\underbrace{f_x \;\; f_y}_{\text{at node 1}} \;\; \underbrace{f_x \;\; f_y}_{\text{at node 2}} \;\; \underbrace{f_x \;\; f_y}_{\text{at node 3.}}\right]$$

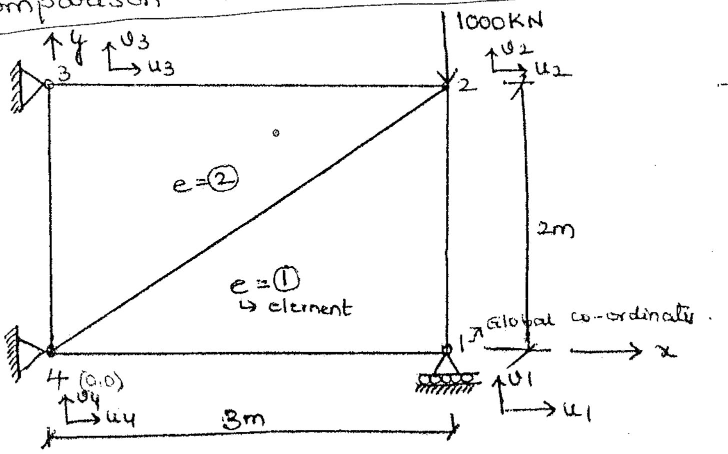
(ii) Surface Traction:-



Node (1) does not come in to picture

Node (2) & Node (3) will get half & half

Q/ Problem: For the two-dimensional loaded plate shown in fig., determine the displacements of nodes 1 and 2 and the element stresses using plane stress conditions. Body force may be neglected in comparison with the external forces.



Thickness, $t = 0.5m$

$E = 30 \times 10^6\ KN/m^2$

$\nu = 0.25$

Sol: For plane stress conditions, the material property matrix is given by

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$D = \frac{30 \times 10^6}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$D = 3.2 \times 10^7 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

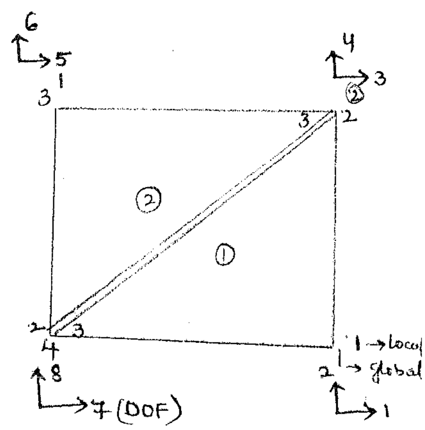
$$D = \begin{bmatrix} 3.2 \times 10^7 & 0.8 \times 10^7 & 0 \\ 0.8 \times 10^7 & 3.2 \times 10^7 & 0 \\ 0 & 0 & 1.2 \times 10^7 \end{bmatrix}$$

Nodal co-ordinates:-

Node no. (Global co-ordinates)	x	y
1	3	0
2	3	2
3	0	2
4	0	0

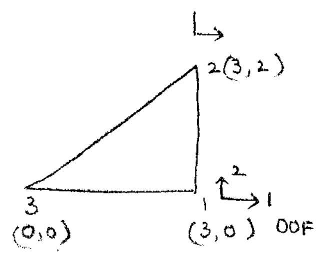
Element connectivity :-

Element No.	1	2	3	(Local co-ordinates)
①	1	2	4	
②	3	4	2	



For Element ①:-

Stiffness, $K = t\ B^T D B A$.



$$\text{Det } J' = \begin{vmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{vmatrix}$$

$$= \begin{vmatrix} 3-0 & 0-0 \\ 3-0 & 2-0 \end{vmatrix}$$

$$\text{Det } J' = \begin{vmatrix} 3 & 0 \\ 3 & 2 \end{vmatrix} = 6$$

$$B^1 = \frac{1}{\text{Det } J^1}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$y_{23} \rightarrow$ At Local coord '2' and 3 $\Rightarrow y_2 - y$ (Glob)

$\Rightarrow 2 - 0$

$= 2$

$y_{31} = y_4 - y_1$

$= 0 - 0 = 0$

$x_{32} = x_4 - x_2$

$= 0 - 3 = -3$

$x_{13} = x_1 - x_4$

$= 3 - 0 = 3$

$$B^1 = \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}_{3\times 6}$$

Then the matrix multiplication,

$$DB^1 = \begin{bmatrix} 3.2\times10^7 & 0.8\times10^7 & 0 \\ 0.8\times10^7 & 3.2\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{bmatrix}_{3\times3}\begin{bmatrix} 1/3 & 0 & 0 & 0 & -1/3 & 0 \\ 0 & -1/2 & 0 & 1/2 & 0 & 0 \\ -1/2 & 1/3 & 1/2 & 0 & 0 & -1/3 \end{bmatrix}_{3\times6}$$

$$DB^1 = 10^7\begin{bmatrix} 1.067 & -0.4 & 0 & 0.4 & -1.067 & 0 \\ 0.267 & -1.6 & 0 & 1.6 & -0.267 & 0 \\ -0.6 & 0.4 & 0.6 & 0 & 0 & -0.4 \end{bmatrix}_{3\times6}$$

$$B^{1T} D B^1 = \frac{1}{6}\begin{bmatrix} 2 & 0 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{6\times3} \times 10^7 \begin{bmatrix} 1.067 & -0.4 & 0 & 0.4 & -1.067 & 0 \\ 0.267 & -1.6 & 0 & 1.6 & -0.267 & 0 \\ -0.6 & 0.4 & 0.6 & 0 & 0 & -0.4 \end{bmatrix}_{3\times6}$$

$$B^{1T} D B^1 = \frac{10^7}{6} \begin{bmatrix} 3.934 & -2.000 & -1.80 & 0.80 & -2.134 & 1.20 \\ -2.001 & 5.600 & 1.20 & -4.80 & 0.801 & -0.80 \\ -1.80 & 1.200 & 1.80 & 0 & 0 & -1.20 \\ 0.801 & -4.80 & 0 & 4.80 & -0.801 & 0 \\ -2.134 & 0.80 & 0 & -0.80 & 2.134 & 0 \\ 1.200 & -0.80 & -1.20 & 0 & 0 & 0.80 \end{bmatrix}_{6\times 6}$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & 3 & 0 \\ 1 & 3 & 2 \\ 1 & 0 & 0 \end{vmatrix} = \frac{1}{2}\{6 - 0\} = \underline{\underline{3}}$$

$\therefore$ Stiffness matrix for Element ①,

$$K^1 = t_e A_e B^{1T} D B^1$$

$$= 0.5 \times 3 \times \frac{10^7}{6} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} 3.934 & -2.00 & -1.80 & 0.80 & -2.134 & 1.20 \\ -2.001 & 5.60 & 1.20 & -4.80 & 0.801 & -0.80 \\ -1.80 & 1.20 & 1.80 & 0 & 0 & -1.20 \\ 0.801 & -4.80 & 0 & +4.80 & -0.801 & 0 \\ -2.134 & 0.80 & 0 & -0.80 & 2.134 & 0 \\ 1.200 & -0.80 & -1.20 & 0 & 0 & 0.80 \end{bmatrix}_{6\times 6} \end{array}$$

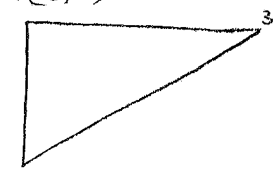
$$K^1 = 10^7 \begin{array}{c} \\ \end{array}\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 7 & 8 & \\ \end{array}$$

$$K^1 = 10^7 \left[\begin{array}{cccccc} 0.9835 & -0.50 & -0.450 & 0.20 & -0.5335 & 0.30 \\ -0.500 & 1.40 & 0.30 & -1.20 & 0.20 & -0.20 \\ -0.450 & 0.30 & 0.45 & 0 & 0 & -0.30 \\ 0.200 & -1.20 & 0 & 1.20 & -0.20 & 0 \\ -0.5335 & 0.20 & 0 & -0.20 & 0.5335 & 0 \\ 0.300 & -0.20 & -0.30 & 0 & 0 & 0.2 \end{array}\right]_{6\times 6} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 7 \\ 8 \end{array}$$

→ Symmetric

For Element ②:-

$$\text{Det } J^1 = \begin{vmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{vmatrix}$$



$$= \begin{vmatrix} 0-3 & 2-2 \\ 0-3 & 0-2 \end{vmatrix} = 3\times 2 = \underline{\underline{6}}$$

$$B^2 = \frac{1}{\text{Det } J^2} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$B^2 = \frac{1}{6} \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}_{3\times 6}$$

$$y_{23} = y_4 - y_2$$
$$= 0-2 = -2$$

Then the matrix multiplication,

$$DB^2 = \begin{bmatrix} 3.2\times10^7 & 0.8\times10^7 & 0 \\ 0.8\times10^7 & 3.2\times10^7 & 0 \\ 0 & 0 & 1.2\times10^7 \end{bmatrix}_{3\times3} \begin{bmatrix} -1/3 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 0 \\ 1/2 & -1/3 & -1/2 & 0 & 0 & 1/3 \end{bmatrix}_{3\times6}$$

$$DB^2 = 10^7 \begin{bmatrix} -1.067 & 0.40 & 0 & -0.4 & 1.067 & 0 \\ -0.267 & 1.60 & 0 & -1.60 & 0.267 & 0 \\ 0.600 & -0.40 & -0.60 & 0 & 0 & 0.4 \end{bmatrix}_{3\times6}$$

$$B^{2^T} D B^2 = \frac{10^7}{6} \begin{bmatrix} -2 & 0 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{6\times3} \begin{bmatrix} -1.067 & 0.40 & 0 & -0.4 & 1.067 & 0 \\ -0.267 & 1.60 & 0 & -1.60 & 0.267 & 0 \\ 0.600 & -0.40 & -0.60 & 0 & 0 & 0.4 \end{bmatrix}_{3\times6}$$

$$= \frac{10^7}{6} \begin{bmatrix} 3.934 & -2.000 & -1.80 & 0.80 & -2.134 & 1.20 \\ -2.001 & 5.600 & 1.20 & -4.80 & 0.801 & -0.80 \\ -1.800 & 1.200 & 1.80 & 0 & 0 & -1.20 \\ 0.801 & -4.80 & 0 & 4.80 & -0.801 & 0 \\ -2.134 & 0.80 & 0 & -0.80 & 2.134 & 0 \\ 1.200 & -0.80 & -1.20 & 0 & 0 & 0.80 \end{bmatrix}_{6\times6}$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 1 & 3 & 2 \end{vmatrix}$$

$$A = \frac{2 \times 3}{2} = \underline{\underline{3}}$$

Stiffness matrix ~~for~~ Element (2),

$$K^2 = t_e A_e \, B^{2^T} D \, B^2$$

$$= \frac{0.5 \times 3 \times 10^7}{6} \begin{array}{c} \begin{array}{cccccc} 5 & 6 & 7 & 8 & 3 & 4 \end{array} \\ \left[\begin{array}{cccccc} 3.934 & -2.000 & -1.80 & 0.80 & -2.134 & 1.20 \\ -2.001 & 5.60 & 1.20 & -4.80 & 0.801 & -0.8 \\ -1.80 & 1.20 & 1.80 & 0 & 0 & -1.2 \\ 0.801 & -4.80 & 0 & 4.80 & -0.801 & 0 \\ -2.134 & 0.80 & 0 & -0.80 & 2.134 & 0 \\ 1.200 & -0.80 & -1.20 & 0 & 0 & 0.8 \end{array}\right] \end{array}$$

6x

$$
K^2 = 10^7 \begin{array}{c} \begin{array}{cccccc} 5 & 6 & 7 & 8 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.9835 & -0.50 & -0.45 & 0.20 & -0.5335 & 0.30 \\ -0.500 & 1.40 & 0.30 & -1.20 & 0.20 & -0.20 \\ -0.450 & 0.30 & 0.45 & 0 & 0 & -0.30 \\ 0.200 & -1.20 & 0 & 1.20 & -0.20 & 0 \\ -0.5335 & 0.20 & 0 & -0.20 & 0.5335 & 0 \\ 0.30 & -0.20 & -0.30 & 0 & 0 & 0.20 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 3 \\ 4 \end{array}
$$

Symmetric.

In the above element matrices, the global dof association is shown on top.

We have v_1, u_3, v_3, u_4, v_4 are all zeros.

∴ Using the elimination approach, the stiffness associated with Degrees of freedom u_1, u_2 and v_2 are:

$$
K = 10^7 \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0.9835 & -0.50 & -0.45 & 0.20 & 0 & 0 & -0.5335 & 0.30 \\ -0.50 & 1.40 & -0.30 & -1.20 & 0 & 0 & 0.20 & -0.20 \\ -0.45 & 0.30 & \begin{array}{c} 0.45 + \\ 0.5335 = 0.9835 \end{array} & 0 & -0.5335 & 0.20 & 0 & -0.50 \\ 0.200 & -1.20 & 0 & \begin{array}{c} 1.20 + \\ 0.2 = 1.40 \end{array} & 0.30 & -0.20 & -0.50 & 0 \\ 0 & 0 & 0 & 0 & 0.9835 & -0.50 & -0.45 & 0.20 \\ 0 & 0 & 0 & 0 & -0.50 & 1.40 & 0.30 & -1.20 \\ -0.5335 & 0.20 & 0 & 0.20 & -0.45 & 0.30 & 0.9835 & 0 \\ 0.300 & -0.20 & -0.30 & 0 & 0.20 & -1.20 & 0 & 1.40 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$

Therefore,

$$[K] = 10^7 \begin{bmatrix} 0.9835 & -0.425 & 0.20 \\ -0.45 & 0.9835 & 0 \\ 0.20 & 0 & 1.40 \end{bmatrix}_{3\times 3}$$

But we know that $[K][U] = [F]$

ie,

$$10^7 \begin{bmatrix} 0.9835 & -0.425 & 0.20 \\ -0.45 & 0.9835 & 0 \\ 0.20 & 0 & 1.40 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1000 \end{bmatrix}$$

The set of Equations given by as,

$$0.9835 u_1 - 0.425 u_2 + 0.20 v_2 = 0.$$

$$-0.45 u_1 + 0.9835 u_2 + 0.v_2 = 0$$

$$0.20 u_1 + 0 u_2 + 1.40 v_2 = -1000 \times 10^{-7}$$

The required values of displacements are

$$\boxed{\begin{aligned} u_1 &= 1.913 \times 10^{-5} \text{ m} \\ u_2 &= 0.87 \times 10^{-5} \text{ m} \\ v_2 &= -7.42 \times 10^{-5} \text{ m} \end{aligned}}$$

For element 1, the element nodal displacement vector is given by

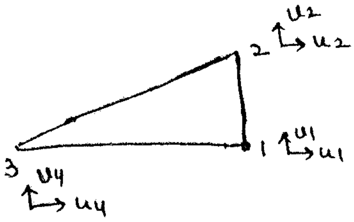
$$U^1 = 10^{-5} \left[1.913, 0, 0.87, -7.42, 0, 0 \right]^T$$

The element stresses σ^1 are calculated from DB^1U^1 as

$$\sigma^1 = DB^1U^1$$

$$\bar{\sigma} = \bar{D}\bar{\epsilon}$$
$$= \bar{D}\{\bar{B}\bar{U}\}$$

$$\sigma^1 = 10^7 \begin{bmatrix} 1.067 & -0.4 & 0 & 0.4 & -1.067 & 0 \\ 0.267 & -1.6 & 0 & 1.6 & -0.267 & 0 \\ -0.6 & 0.40 & 0.6 & 0 & 0 & -0.4 \end{bmatrix}_{3\times6} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_4 \\ v_4 \end{bmatrix}_{6\times}$$



$$\sigma^1 = 10^7 \times 10^{-5} \begin{bmatrix} 1.067 & -0.4 & 0 & 0.4 & -1.067 & 0 \\ 0.267 & -1.6 & 0 & 1.6 & -0.267 & 0 \\ -0.6 & 0.40 & 0.6 & 0 & 0 & -0.4 \end{bmatrix}_{3\times6} \begin{bmatrix} 1.913 \\ 0 \\ 0.87 \\ -7.42 \\ 0 \\ 0 \end{bmatrix}_{6\times1}$$

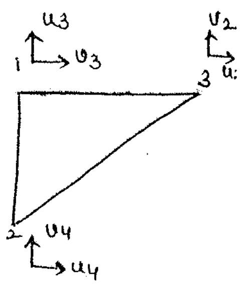
$$\sigma^1 = 10^2 \begin{bmatrix} -0.9269 \\ -11.362 \\ -0.6258 \end{bmatrix} = \begin{bmatrix} -92.70 \\ -1136.20 \\ -62.58 \end{bmatrix}$$

$$\sigma^1 = [-92.70, \ -1136.20, \ -62.58]^T \ \text{KN/m}^2$$

Similarly, For Element ②,

$$U^2 = [u_3, v_3, u_4, v_4, u_2, v_2]^T$$

$$U^2 = 10^{-5}[0, 0, 0, 0, 0.87, -7.42]^T$$



$$\sigma^2 = \bar{D}\,\bar{E}$$

$$= \bar{D}(B^2 U^2)$$

$$\sigma^2 = (\bar{D} B^2)\, U^2$$

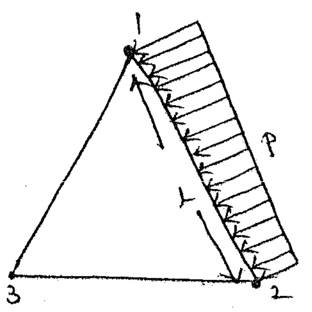
$$\sigma^2 = \frac{10^7}{10^5} \times \begin{bmatrix} -1.067 & 0.40 & 0 & -0.4 & 1.067 & 0 \\ -0.267 & 1.60 & 0 & -1.60 & 0.267 & 0 \\ 0.600 & -0.40 & -0.60 & 0 & 0 & 0.4 \end{bmatrix}_{3\times 6} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.87 \\ -7.42 \end{bmatrix}_{6\times 1}$$

$$\sigma^2 = 100 \begin{bmatrix} 0.9283 \\ 0.2330 \\ -2.968 \end{bmatrix} = \begin{bmatrix} 92.83 \\ 23.30 \\ -296.80 \end{bmatrix} \ KN/m^2$$

$$\boxed{\sigma^2 = [92.83,\ 23.30,\ -296.80]^T \ KN/m^2}$$

Note: Loading!

(i)



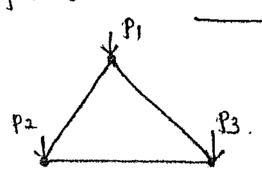
$$P_1 = \frac{Pl}{2}$$

$$P_2 = \frac{Pl}{2}$$

$$P_3 = 0.$$

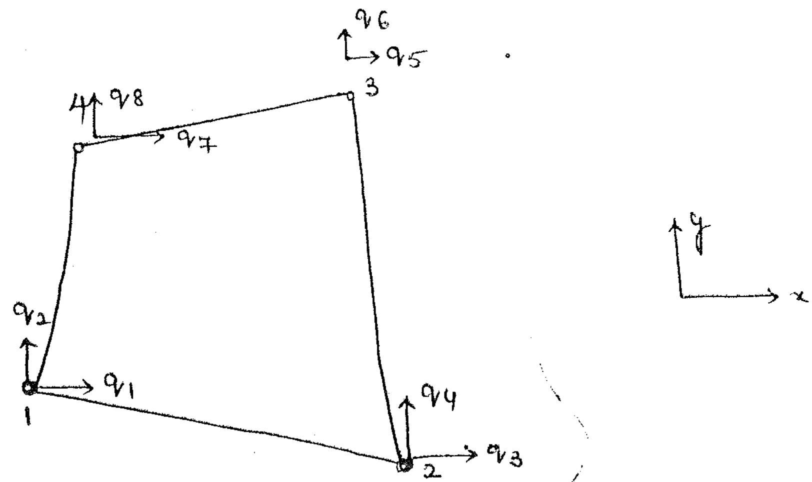
(ii) If density material & vol of the plate, is given,

then the Weight of the plate $= \dfrac{\text{density} \times \text{vol. of the plate}}{3}$



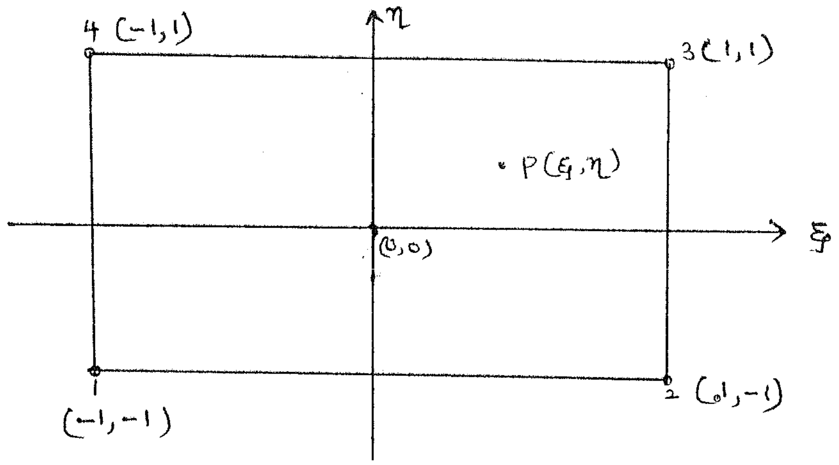
Derive the B-Matrix of Four-Noded Quadrilateral Elements (49)

Four Noded Isoparametric Quadrilateral Element :-



FOUR NODED QUADRILATERAL ELEMENT

Each node has got two DOF.



THE QUADRILATERAL ELEMENT IN ξ, η SPACE (The Master Element)

No. of shape functions = 4

The General expression for shape function,

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$$

$$\therefore \quad N_1 = \frac{1}{4}(1-\xi)(1-\eta) \text{ corresponds to } (\xi_i,\eta_i) = (-1,-1)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Check:

N_1 at $1 = \frac{1}{4} \times 2 \times 2 = 1$

N_1 at $2, 3, 4 = 0$

N_2 at $2 = 1$

N_2 at $1, 3, 4 = 0$.

N_3 at $3 = 1$

N_3 at $1, 2, 4 = 0$

N_4 at $4 = 1$

N_4 at $1, 2, 3 = 0$.

We now express the displacement field within the element in terms of the nodal values.

$$U = N_1 U_1 + N_2 U_2 + N_3 U_3 + N_4 U_4$$

$$V = N_1 V_1 + N_2 V_2 + N_3 V_3 + N_4 V_4$$

Since in Isoparametric formulation,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

Strains:-

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta}$$

(or)

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = [J] \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

where 'J' is the Jacobian matrix.

The elements of 'J' matrix are

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$J_{12} = \frac{\partial y}{\partial \xi} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \frac{1}{4}\left[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right]$$

$$\therefore \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

or

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \underbrace{\frac{1}{\det J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}}_{J^{-1}} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} \longrightarrow (1)$$

These expressions will be used in the derivation of the element stiffness matrix. An additional result that will be needed is the relation,

$$\boxed{dx \cdot dy = \det J \cdot d\xi \cdot d\eta}$$

Element stiffness Matrix:

The stiffness matrix for the quadrilateral element can be derived from the strain Energy in the body; given by

$$\bar{U} = \int_V \frac{1}{2}\sigma^T \epsilon \, dV$$

$$= \sum_e t_e \int \frac{1}{2}\sigma^T \epsilon . dA$$

where t_e = thickness of element 'e'.

The strain -- displacement relations are.

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$

similar to Equation ①, we can write

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{\det J}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \longrightarrow ②$$

From Equations ① and ②, we get

$$\epsilon = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \frac{1}{\det J.}\begin{bmatrix} J_{22}\frac{\partial u}{\partial \xi} - J_{12}\frac{\partial u}{\partial \eta} \\ -J_{21}\frac{\partial v}{\partial \xi} + J_{11}\frac{\partial v}{\partial \eta} \\ -J_{21}\frac{\partial u}{\partial \xi} + J_{11}\frac{\partial u}{\partial \eta} + J_{22}\frac{\partial v}{\partial \xi} - J_{12}\frac{\partial v}{\partial \eta} \end{bmatrix}_{3\times}$$

$$\epsilon = \frac{1}{\det J}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \underset{(\text{Say})}{[A]}\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix}$$

We know that,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$= \frac{1}{4}\Big[(1-\xi)(1-\eta)\,u_1 + (1+\xi)(1-\eta)\,u_2 + (1+\xi)(1+\eta)\,u_3 + (1-\xi)(1+\eta)\,u_4\Big]$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

$$= \frac{1}{4}\Big[(1-\xi)(1-\eta)\,v_1 + (1+\xi)(1-\eta)\,v_2 + (1+\xi)(1+\eta)\,v_3 + (1-\xi)(1+\eta)\,v_4\Big]$$

$$\frac{\partial u}{\partial \xi} = \frac{1}{4}\Big[-(1-\eta)u_1 + (1-\eta)u_2 + (1+\eta)u_3 + \{-(1+\eta)\}u_4\Big]$$

$$\frac{\partial u}{\partial \eta} = \frac{1}{4}\Big[-(1-\xi)u_1 - (1+\xi)u_2 + (1+\xi)u_3 + (1-\xi)u_4\Big]$$

$$\frac{\partial v}{\partial \xi} = \frac{1}{4}\Big[-(1-\eta)v_1 + (1-\eta)v_2 + (1+\eta)v_3 + [-(1+\eta)]v_4\Big]$$

$$\frac{\partial v}{\partial \eta} = \frac{1}{4}\Big[-(1-\xi)v_1 - (1+\xi)v_2 + (1+\xi)v_3 + (1-\xi)v_4\Big]$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$= [G][U] \,. \qquad \text{But } \bar{\epsilon} = \bar{B}\bar{U}$$

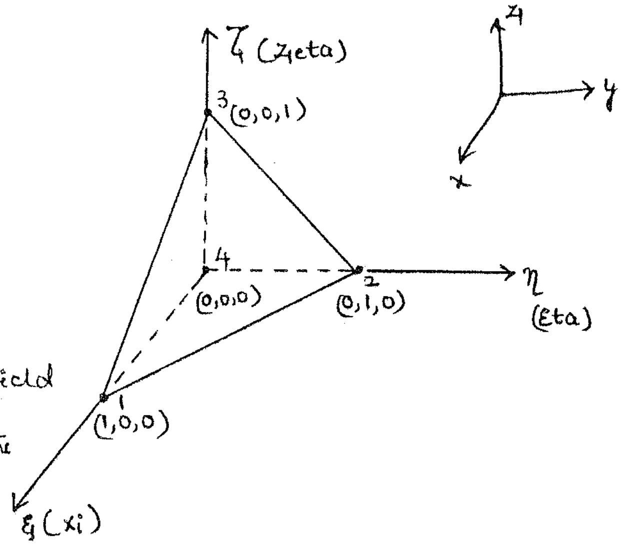
$$\therefore \boxed{B = AG}$$

and Also stress, $\sigma = DBU$

$$\boxed{k_e = t\int^{+1}\!\!\int^{+1} B^T D B\, |J|\, d\xi\, d\eta}$$

ISOPARAMETRIC ELEMENTS FOR 3D ELEMENT :

FOUR NODED TETRAHEDRON :-



$N_1 = \xi$

$N_2 = \eta$

$N_3 = \zeta$

$N_4 = 1 - \xi - \eta - \zeta$

We now express the displacement field with in the element in terms of the nodal values.

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3 + N_4 w_4$$

Since in Isoparametric formulation,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$z = N_1 z_1 + N_2 z_2 + N_3 z_3 + N_4 z_4$$

Strains:

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial \eta}$$

$$\frac{\partial u}{\partial \zeta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \zeta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \zeta} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial \zeta}$$

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix} \longrightarrow (1)$$

where, $J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}_{3\times 3}$

$$J = \begin{bmatrix} x_{14} & y_{14} & z_{14} \\ x_{24} & y_{24} & z_{24} \\ x_{34} & y_{34} & z_{34} \end{bmatrix}$$

$$\det J = \begin{vmatrix} x_{14} & y_{14} & z_{14} \\ x_{24} & y_{24} & z_{24} \\ x_{34} & y_{34} & z_{34} \end{vmatrix}$$

The volume of the element is given by

$$V_e = \frac{1}{6} |\det J|$$

The inverse relation corresponding to equat 'J' is given by

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} \end{bmatrix}$$

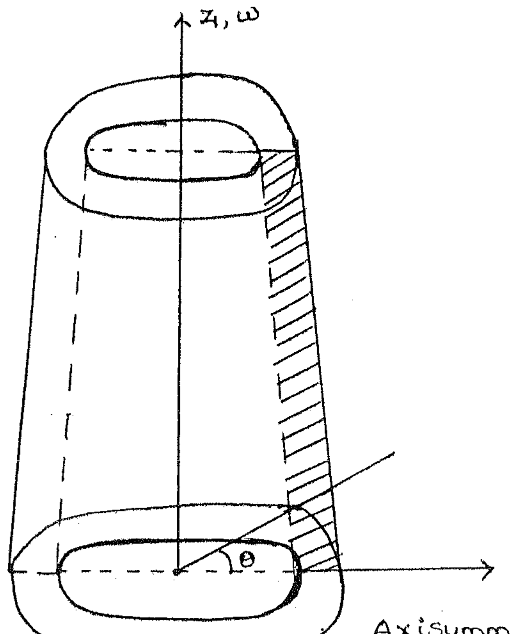
$\epsilon = B \cdot q$ where 'B' is a 6×12 matrix.

The element stiffness matrix is given by

$$\boxed{K^e = V_e B^T D B.}$$

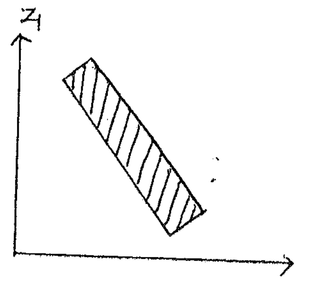
(4) Give the Finite Element Formulation of axisymmetric triangular element. Derive the B-matrix for axisymmetric 3-noded triangular element.

Sol: Axi-Symmetric Formulation:-



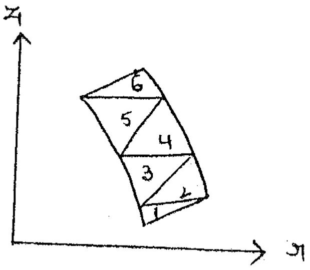
Axisymmetric problem.

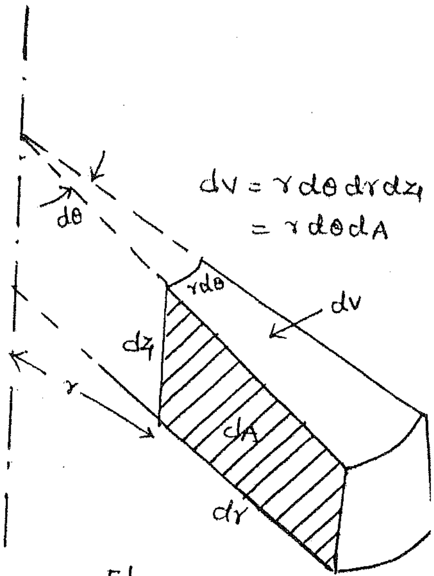
Instead of x, y, here comes r & z.



Become a 2D problem.

We discritize the c/s instead of the entire object





Elemental Volume.

considering the elemental volume as shown in fig, the potential energy can be written in the form,

$$\Pi = \frac{1}{2}\int_0^{2\pi}\int_A \sigma^T \epsilon_r \, dA \, d\theta - \int_0^{2\pi}\int_A u^T f_r \, dA \, d\theta - \int_0^{2\pi}\int_L u^T T_r \, dl \, d\theta - \sum_i u_i^T P_i \longrightarrow (1)$$

where $r\,dl\,d\theta$ is the elemental surface area and the point load 'P_i' represents a line load distributed around a circle as shown in fig.

All variables in the integrals are independent of 'θ'.

Thus Equation ① can be written as

$$\Pi = 2\pi\left[\frac{1}{2}\int_A \sigma^T \epsilon_r\, dA - \int_A U^T f_r\, dA - \int_L U^T T_r\, dl\right] - \sum_i u_i^T P_i \rightarrow ②$$

where $\bar{u} = [u, w]^T$

$\bar{f} = [f_r, f_{z_1}]^T$

$\bar{T} = [T_r, T_{z_1}]^T$

Relationship between strains 'ϵ' and displacements 'u' as

$$\epsilon = [\epsilon_r, \epsilon_{z_1}, \gamma_{rz_1}, \epsilon_\theta]^T$$

$$= \left[\frac{\partial u}{\partial r}, \frac{\partial w}{\partial z_1}, \frac{\partial u}{\partial z_1} + \frac{\partial w}{\partial r}, \frac{u}{r}\right]^T$$

The stress vector is correspondingly defined as,

$$\bar{\sigma} = [\sigma_r, \sigma_{z_1}, \tau_{rz_1}, \sigma_\theta]^T$$

The stress - strain relations are given in the usual form,

$$\bar{\sigma} = \bar{D}\bar{\epsilon}$$

where the (4x4) matrix 'D' can be written as,

$$D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}_{4\times4}$$

In the Galerkin formulation, we require

$$2\pi \int_A \sigma^T \epsilon(\phi)\, r\, dA - \left(2\pi \int_A \phi^T f\, r\, dA + 2\pi \int_L \phi^T T\, r\, dl + \Sigma \phi_i^T P_i \right) = 0$$

where $\phi = [\phi_r, \phi_z]^T$

$$\epsilon(\phi) = \left[\frac{\partial \phi_r}{\partial r}, \frac{\partial \phi_z}{\partial z}, \frac{\partial \phi_r}{\partial z} + \frac{\partial \phi_z}{\partial r}, \frac{\phi_r}{r} \right]^T$$

Stiffness, $K = \int_V B^T D B \cdot dV$

$dV = A \cdot dx$ for 1D - problem

$dV = t \cdot dA$ for 2D - problem.

** FINITE ELEMENT MODELLING USING TRIANGULAR ELEMENT:

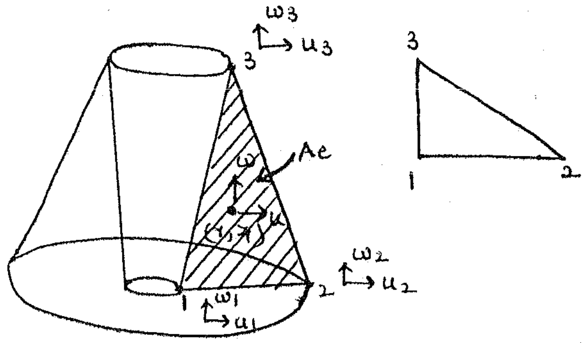
The two-dimensional region defined by the revolving area is divided into triangular elements.

Using the three shape functions N_1, N_2 and N_3;

$N_1 = \xi$

$N_2 = \eta$

$N_3 = 1 - \xi - \eta$



Displacements:

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3$$

co-ordinates:-

$$r = N_1 r_1 + N_2 r_2 + N_3 r_3$$

$$z = N_1 z_1 + N_2 z_2 + N_3 z_3$$

By using shape functions values, the displacements are:

$$u = \xi u_1 + \eta u_2 + (1-\xi-\eta) u_3$$

$$v = \xi v_1 + \eta v_2 + (1-\xi-\eta) v_3$$

By using the Isoparametric representation, we find

$$r = \xi r_1 + \eta r_2 + (1-\xi-\eta) r_3$$

$$z = \xi z_1 + \eta z_2 + (1-\xi-\eta) z_3$$

The chain rule of differentiation gives,

$$\begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\ \\ \dfrac{\partial u}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \\ \dfrac{\partial u}{\partial z} \end{bmatrix} \longrightarrow (a)$$

and

$$\begin{bmatrix} \dfrac{\partial w}{\partial \xi} \\ \\ \dfrac{\partial w}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \dfrac{\partial w}{\partial r} \\ \\ \dfrac{\partial w}{\partial z} \end{bmatrix} \longrightarrow (b)$$

where the Jacobian is given by

$$J = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix}$$

We have used the notation $r_{ij} = r_i - r_j$ and

$$z_{ij} = z_i - z_j$$

The determinant of 'J' is

$$\det J = r_{13} z_{23} - r_{23} z_{13}.$$

Recall that $|\det J| = 2A_e$. That is, the absolute value of the determinant of 'J' equals twice the area of the element.

The inverse relations for Equations ⓐ and ⓑ are given by

$$\begin{bmatrix} \dfrac{\partial u}{\partial r} \\[2ex] \dfrac{\partial u}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\[2ex] \dfrac{\partial u}{\partial \eta} \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} \dfrac{\partial w}{\partial r} \\[2ex] \dfrac{\partial w}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial w}{\partial \xi} \\[2ex] \dfrac{\partial w}{\partial \eta} \end{bmatrix}$$

where $J^{-1} = \dfrac{1}{\det J} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & r_{13} \end{bmatrix}$

Introducing these transformations relationships in to strain-displacement relations, we get

$$\epsilon = \begin{bmatrix} \dfrac{\partial u}{\partial r} \\[2ex] \dfrac{\partial w}{\partial z} \\[2ex] \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \\[2ex] \dfrac{u}{r} \end{bmatrix} = \begin{bmatrix} \dfrac{z_{23}(u_1-u_3) - z_{13}(u_2-u_3)}{\det J} \\[2ex] \dfrac{-r_{23}(w_1-w_3) + r_{13}(w_2-w_3)}{\det J} \\[2ex] \dfrac{-r_{23}(u_1-u_3) + r_{13}(u_2-u_3) + z_{23}(w_1-w_3) - z_{13}(w_2-w_3)}{\det J} \\[2ex] \dfrac{N_1u_1 + N_2u_2 + N_3u_3}{r} \end{bmatrix}$$

This can be written in the matrix form as,

$$\epsilon = B q$$

where the element strain-displacement matrix, q: displacement vector.

of dimension (4×6), is given by

$$B = \begin{bmatrix} \frac{z_{23}}{\det J} & 0 & \frac{z_{31}}{\det J} & 0 & \frac{z_{12}}{\det J} & 0 \\ 0 & \frac{r_{32}}{\det J} & 0 & \frac{r_{13}}{\det J} & 0 & \frac{r_{21}}{\det J} \\ \frac{r_{32}}{\det J} & \frac{z_{23}}{\det J} & \frac{r_{13}}{\det J} & \frac{z_{31}}{\det J} & \frac{r_{21}}{\det J} & \frac{z_{12}}{\det J} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix}_{4 \times 6}$$

$$dV = r \cdot d\theta \cdot dr$$

$$= (2\pi r)\, A$$

$$K = B^T D B \cdot 2\pi r \cdot A$$

$$A = \frac{1}{2} \det J .$$

Q1 Give the Origin of the Finite Element Method its applications Advantages and Disadvantages. Give examples where necessary.

Sol: **Origin:**

Basic Idea of the FEM originated from advances in aircraft Structural Analysis. ~~Hrenikof~~ Hrenikoff presented a solution of elasticity problems using the frame work method. The term Finite element was first coined and used by Clough in 1960.

Applications:

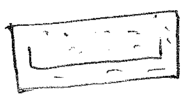
- The Finite element Method has become a powerful tool for the numerical solution of a wide range of engineering problems.
- Applications range from deformation and stress analysis of automotive, aircraft, building and bridge structures to field analysis of heat flux, fluid flow, magnetic flux, seepage and other flow problems.
- With the Advances in computer technology and CAD systems, complex problems can be modelled with relative ease.
- Several alternative configurations can be tested on a computer before the 1st proto-type is built.
- ** In the Finite element Method of Analysis a complex region defining a continuum is discretized into simple geometric shapes called finite elements.

- The material properties and Governering relationships are considered over these elements and expressed in terms of unknown values at element corners.
- An Assembly process considering the loading and constraints results in a set of equations.
- Solution of these equations gives us the approximate behaviour of the Continuum.

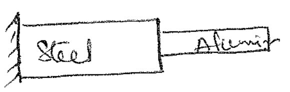
Advantages of FEM:

1. This method can be easily applied to structures of irregular geometry.
2. This method can handle any type of boundary conditions.

prescribed displacements.

3. Material isotropy and inhomogeneity can be treated without much difficulty. Ex RCC 
4. It can handle structures with different materials.

Ex

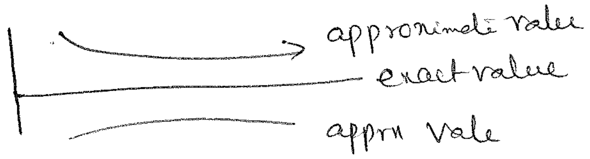


5. It can handle structures with complex loadings.
6. This method is easily amenable to computer programmings.

Dis-Advantages of FEM:

1. The Results of the analysis are mesh dependant.

2. This method is approximate.



$< 10\%$ error

3. Computer is essential for the use of this method.

4. Input data preparation is cumbersome. (PRE PROCESSOR)

5. Too much output information to handle. (POST PROCESSOR),

Q ② Basic Steps Involved in the Finite Element Method:

FEM: Finite Element Method is a Numerical Method

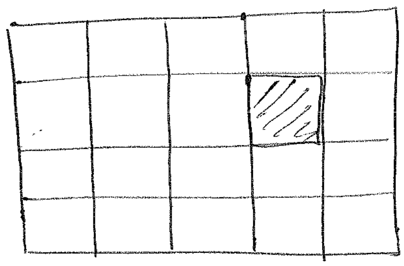
Discretization: In the Discretization the structure is divided into very very small parts.

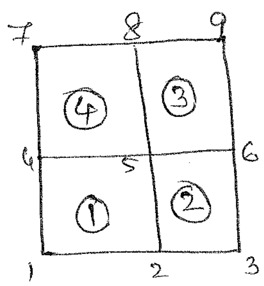
Steps Involved in FEM:

① Idealise the structure — As a plate

② Divide the structure into small elements

Discretization :- is the process which is divided into small elements





The elements which are connected at four corners is called "nodes".

elements : ① ② ③ ④

Nodes : 1 2 3 4 5 6 7 8 9

③ Calculate element stiffness matrix

Element Connection

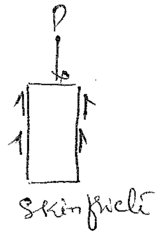
(1)	1	2	5	4
(2)	2	3	6	5
(3)	4	5	8	9
(4)	5	6	9	8

Numbering is to be done by Anticlockwise manner.

For Bar element $[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(4) Assemble the Global stiffness matrix

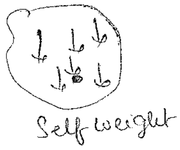
(5) calculate the element loads.



Loading consists of Three types

(a) body Load (or) Force ⟶ Self weight (or) Gravity load.

(b) Traction force (Surface load).



(c) Point Load.

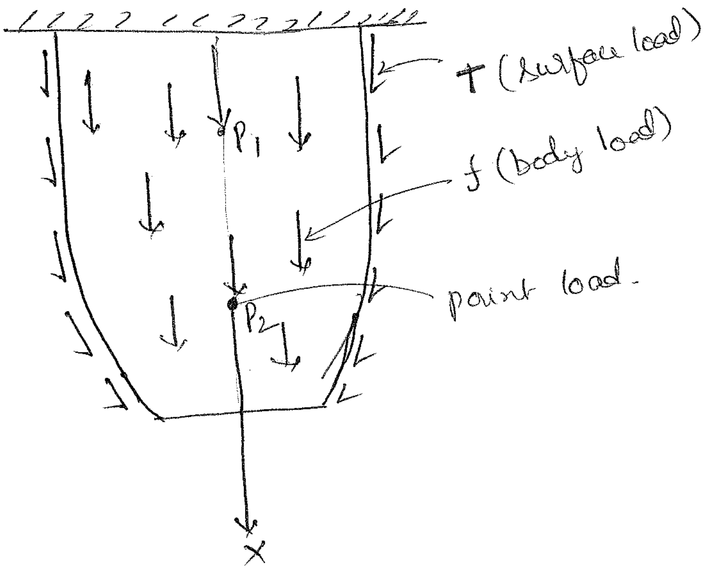
body Force (f): Body force is a distributed force acting on acting on every elemental volume of the body

units: Force/unit volume.

Traction Force(T) Traction force is a distributed load acting on the surface of the body. units: Force/unit area.
Force/unit length

point load (Pi): is a force acting at a point i

$P_i =$ $i = 1, 2, 3, \ldots n$.



Above Figure Showing The one dimensional bar.

(6) Assemble the load Vector, to obtain Global load Vector 'F'

(7) ~~From~~ For The System of equation $KU = F$

U = unknown displacement

(8) Solve The System of equation (ie) Gauss elimination Method.

RAYLEIGH – RITZ method

The total potential Energy (Π) of an elastic body, is defined as the sume of total strain Energy (U) and the work potential.

Total Energy (Π) = strain Energy (U) + Work Energy (W)

↳ internal energy ↳ external

The total strain energy is given by

$$U = \frac{1}{2}\int \sigma\, \epsilon \cdot dv$$

the Work energy (WP) is given by

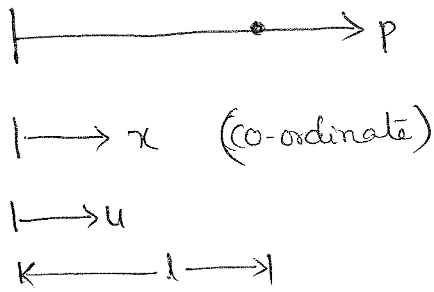
$$WP = -\int u \cdot f \cdot dv - \int u \cdot T \cdot ds - \sum_{i=1}^{n} P_i U_i$$

Total potential for General elastic body.

$$\Pi = \frac{1}{2}\int \sigma\, \epsilon \cdot dv - \int u \cdot f \cdot dv - \int u \cdot T \cdot ds - \sum_{i=1}^{n} P_i U_i$$

where f = body force.
T = surface force.
U = displacement.

For one dimensional Element:



Shress (σ) = Strain $\times$ E $\qquad\qquad \epsilon = \frac{\partial u}{\partial x}$

$$\sigma = E \times \frac{\partial u}{\partial x}$$

$$\therefore \; U = \frac{1}{2} \int E \left(\frac{du}{dx}\right)^2 . dv$$

$$= \frac{1}{2} \int E \left(\frac{du}{dx}\right)^2 . A . dx.$$

$$= \frac{1}{2} EA \int_0^l \left(\frac{\partial u}{dx}\right)^2 dx$$

$$u = L_1 + L_2 x + L_3 x^2$$

$L_1 \; L_2 \; L_3$ are are constants.

By substituting in eqn (1) we get

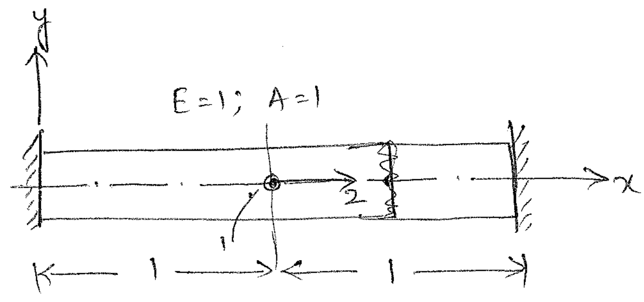
$$\Pi = \frac{1}{2} EA \int_0^l \left(\frac{du}{dx}\right)^2 dx - \int f \times u \, dv - \int T . U . dA - \sum_{i=1}^{n} P_i U_i$$

By using the principle of Minimum potential energy

ie. $\dfrac{\partial \Pi}{\partial (L_1 \cdots L_n)} = 0$ find unknowns $L_1, L_2 \cdots L_n$.

Ex: 1-2
34 page

Use the Rayleigh Ritz method to find the displacement of the mid point of the Rod shown in fig below.



Sol:

The potential energy for the linear elastic one dimensional rod when body force is neglected.

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - 2U_1$$

where $U_1 = u \quad (x=1)$

Let us consider a polynomial function

U_1 = deflection at point (2).

$$U = a_1 + a_2 x + a_3 x^2$$

@ $x = 0$; $u = 0$ $\Rightarrow$ $0 = a_1$

@ $x = 2$; $u = 0$ $\Rightarrow$ $0 = a_1 + 2a_2 + 4a_3$

~~Thus~~

Hence

$$2a_2 = -4a_3.$$

$$\boxed{a_2 = -2a_3}$$

$$U = a_1 + a_2 x + a_3 x^2$$

Substitute
$a_1 = 0$
$a_2 = -2a_3$.

$$= 0 + (-2a_3 \cdot x) + a_3 x^2$$

$$= -2a_3 \cdot x + a_3 \cdot x^2$$

$$= a_3(-2x + x^2)$$

$u = a_3(-2x + x^2)$

at $x = 1$ $\quad u = a_3(-2x + x^2)$

$U = -a_3.$

$\boxed{U_1 = -a_3.}$

Then $\dfrac{du}{dx} = a_3.(-2 + 2x)$

$= -2a_3(-1 + x)$

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - 2U_1$$

$$= \frac{1}{2}EA\int_0^2 4a_3^2(-1+x)^2 dx - 2(-a_3)$$

$$= 2a_3^2\int_0^2 (1 - 2x + x^2)\,dx + 2a_3$$

$$= 2a_3^2\int_0^2 (x^2 - 2x + 1)\,dx + 2a_3$$

$$= 2a_3^2 \left[\left(\frac{x^3}{3} - \frac{2x^2}{2} + x\right)\right]_0^2 + 2a_3$$

$$= 2a_3^2\left[\frac{8}{3} - 4 + 2\right] + 2a_3.$$

$$= 2a_3^2\left[\frac{8}{3} - 2\right] + 2a_3 \implies \frac{16}{3}a_3^2 - 4a_3^2 + 2a_3.$$

$$\Pi = \frac{16}{3}a_3^2 - 4a_3^2 + 2a_3$$

$\therefore \dfrac{\partial\Pi}{\partial a_3} = 0$

$$\frac{\partial\Pi}{\partial a_3} = \frac{32}{3}a_3 - 8a_3 + 2 = 0$$

$2.667\, a_3 = -2$

$a_3 = -0.75$

$U_1 = -a_3 \implies 0.75$

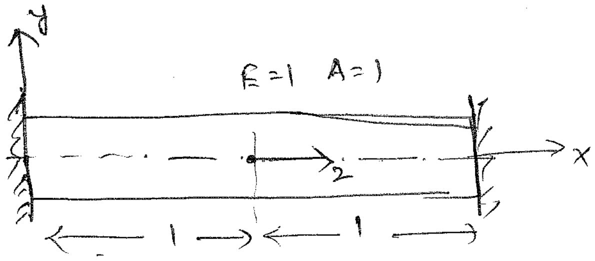
The stress in the bar is given by $\sigma = E.\epsilon$

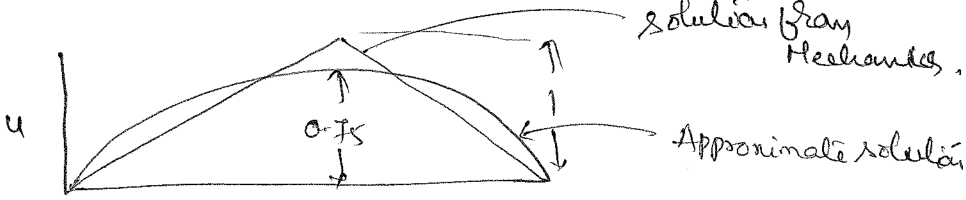
$= E.\dfrac{\partial u}{dx} = 2a_3.(-1+x)$

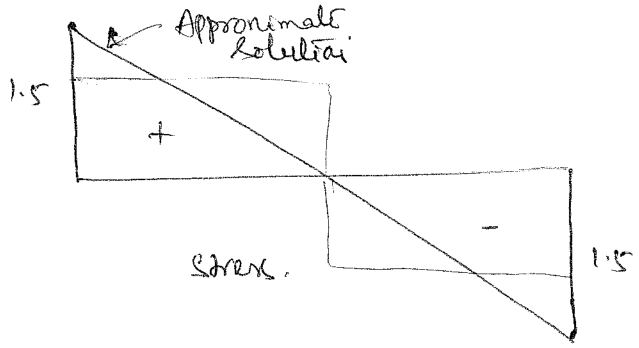
$= -2 \times 0.75(-1+x)$

$= -1.5(-1+x)$

$\boxed{\sigma = 1.5(1-x)}$


y
E=1 A=1
x
2
1
1


solution from Mechanics.
u
0.75
1
Approximate solution


Approximate solution
1.5
+
-
Stress.
1.5

1

(3×3) symmetric matrix. However, we represent stress by the six independent components as in

$$\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}]^{\mathrm{T}} \tag{1.5}$$

where σ_x, σ_y, σ_z are normal stresses and τ_{yz}, τ_{xz}, τ_{xy}, are shear stresses. Let us consider equilibrium of the elemental volume shown in Fig. 1.2. First we get forces on faces by multiplying the stresses by the corresponding areas. Writing $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$ and recognizing $dV = dx\,dy\,dz$, we get the equilibrium equations

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z &= 0
\end{aligned} \tag{1.6}$$

1.5 BOUNDARY CONDITIONS

Referring to Fig. 1.1, we find that there are displacement boundary conditions and surface-loading conditions. If $\mathbf{u}$ is specified on part of the boundary denoted by S_u, we have

$$\mathbf{u} = \mathbf{0} \text{ on } S_u \tag{1.7}$$

We can also consider boundary conditions such as $\mathbf{u} = \mathbf{a}$, where $\mathbf{a}$ is a given displacement.

We now consider the equilibrium of an elemental tetrahedron $ABCD$, shown in Fig. 1.3, where DA, DB, and DC are parallel to the x-, y-, and z-axes, respectively, and area ABC, denoted by dA, lies on the surface. If $\mathbf{n} = [n_x, n_y, n_z]^{\mathrm{T}}$ is the unit normal to dA, then area $BDC = n_x\,dA$, area $ADC = n_y\,dA$, and area $ADB = n_z\,dA$. Consideration of equilibrium along the three axes directions gives

$$\begin{aligned}
\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z &= T_x \\
\tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z &= T_y \\
\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z &= T_z
\end{aligned} \tag{1.8}$$

These conditions must be satisfied on the boundary, S_T, where the tractions are applied. In this description, the point loads must be treated as loads distributed over small, but finite areas.

1.6 STRAIN–DISPLACEMENT RELATIONS

We represent the strains in a vector form that corresponds to the stresses in Eq. 1.5,

$$\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}]^{\mathrm{T}} \tag{1.9}$$

where ϵ_x, ϵ_y, and ϵ_z are normal strains and γ_{yz}, γ_{xz}, and γ_{xy} are the engineering shear strains.

Figure 1.4 gives the deformation of the dx–dy face for small deformations, which we consider here. Also considering other faces, we can write

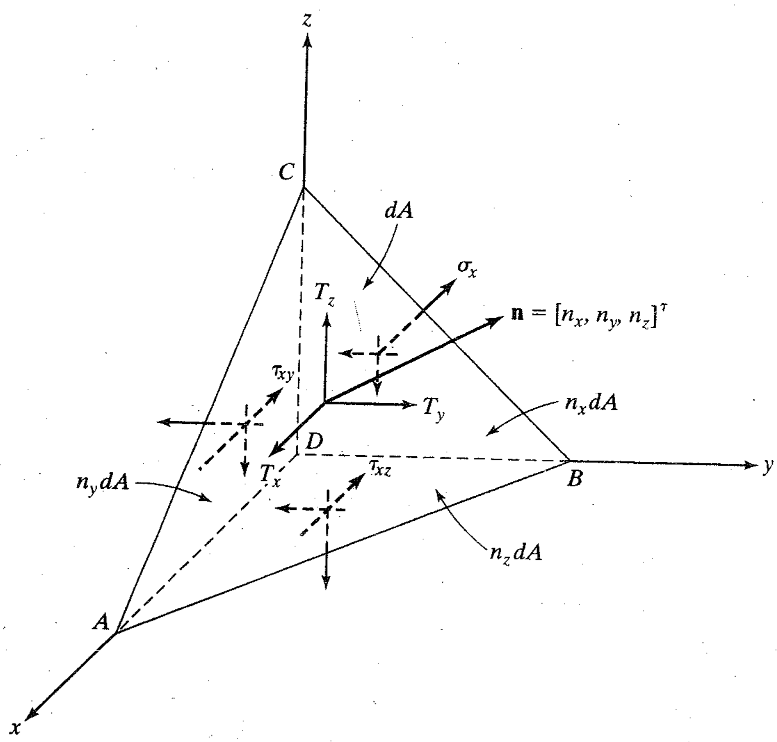


FIGURE 1.3 An elemental volume at surface.

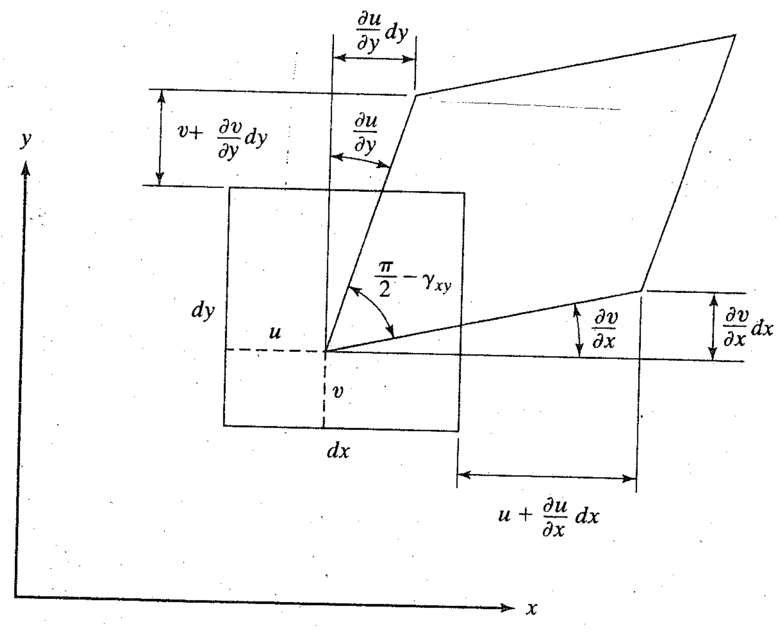


FIGURE 1.4 Deformed elemental surface.

②

$$\boldsymbol{\epsilon} = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^{\mathrm{T}} \tag{1.10}$$

These strain relations hold for small deformations.

1.7 STRESS–STRAIN RELATIONS

For linear elastic materials, the stress–strain relations come from the generalized Hooke's law. For isotropic materials, the two material properties are Young's modulus (or modulus of elasticity) E and Poisson's ratio ν. Considering an elemental cube inside the body, Hooke's law gives

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\
\epsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\
\epsilon_z &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \\
\gamma_{yz} &= \frac{\tau_{yz}}{G} \\
\gamma_{xz} &= \frac{\tau_{xz}}{G} \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned} \tag{1.11}$$

The shear modulus (or modulus of rigidity), G, is given by

$$G = \frac{E}{2(1+\nu)} \tag{1.12}$$

From Hooke's law relationships (Eq. 1.11), note that

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{(1-2\nu)}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{1.13}$$

Substituting for $(\sigma_y + \sigma_z)$ and so on into Eq. 1.11, we get the inverse relations

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \tag{1.14}$$

$\mathbf{D}$ is the symmetric (6×6) material matrix given by

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \tag{1.15}$$

Special Cases

One dimension. In one dimension, we have normal stress σ along x and the corresponding normal strain ϵ. Stress–strain relations (Eq. 1.14) are simply

$$\sigma = E\epsilon \tag{1.16}$$

Two dimensions. In two dimensions, the problems are modeled as plane stress and plane strain.

Plane Stress. A thin planar body subjected to in-plane loading on its edge surface is said to be in plane stress. A ring press fitted on a shaft, Fig. 1.5a, is an example. Here stresses σ_z, τ_{xz}, and τ_{yz} are set as zero. The Hooke's law relations (Eq. 1.11) then give us

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \\
\epsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \\
\gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy}
\end{aligned} \tag{1.17}$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

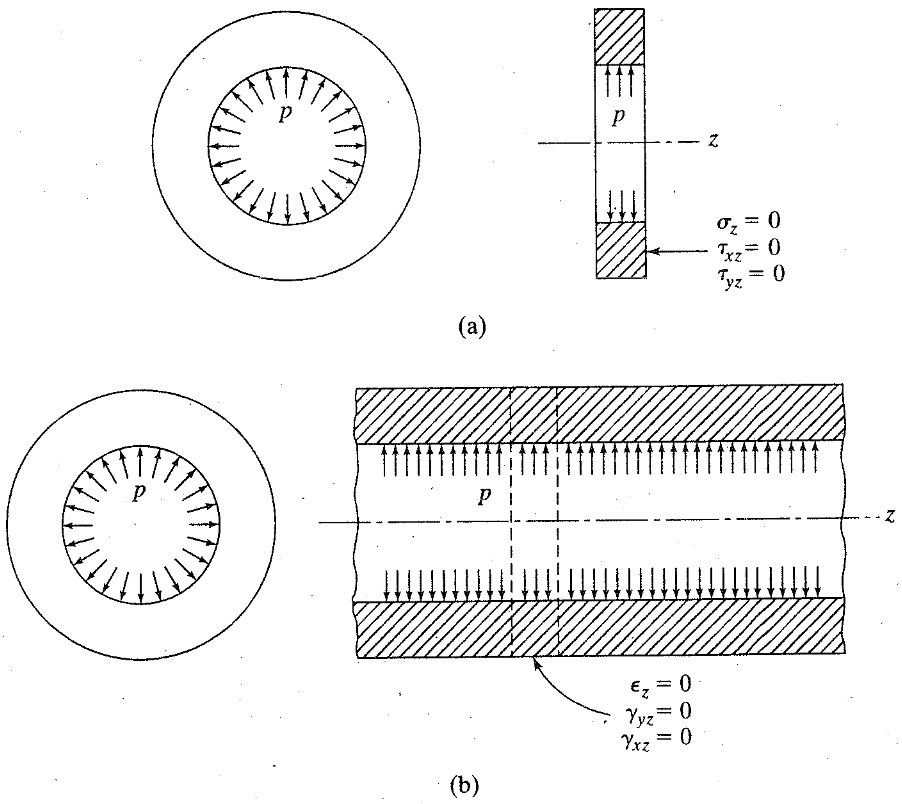


FIGURE 1.5 (a) Plane stress and (b) plane strain.

3

The inverse relations are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{1.18}$$

which is used as $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$.

Plane Strain. If a long body of uniform cross section is subjected to transverse loading along its length, a small thickness in the loaded area, as shown in Fig. 1.5b, can be treated as subjected to plane strain. Here ϵ_z, γ_{zx}, γ_{yz} are taken as zero. Stress σ_z may not be zero in this case. The stress–strain relations can be obtained directly from Eqs. 1.14 and 1.15:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{1.19}$$

D here is a (3×3) matrix, which relates three stresses and three strains.

Anisotropic bodies, with uniform orientation, can be considered by using the appropriate **D** matrix for the material.

1.8 TEMPERATURE EFFECTS

If the temperature rise $\Delta T(x, y, z)$ with respect to the original state is known, then the associated deformation can be considered easily. For isotropic materials, the temperature rise ΔT results in a uniform strain, which depends on the coefficient of linear expansion α of the material. α, which represents the change in length per unit temperature rise, is assumed to be a constant within the range of variation of the temperature. Also, this strain does not cause any stresses when the body is free to deform. The temperature strain is represented as an initial strain:

$$\boldsymbol{\epsilon}_0 = [\alpha\Delta T, \alpha\Delta T, \alpha\Delta T, 0, 0, 0]^{\mathrm{T}} \tag{1.20}$$

The stress–strain relations then become

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) \tag{1.21}$$

In **plane stress**, we have

$$\boldsymbol{\epsilon}_0 = [\alpha\Delta T, \alpha\Delta T, 0]^{\mathrm{T}} \tag{1.22}$$

In **plane strain**, the constraint that $\epsilon_z = 0$ results in a different $\boldsymbol{\epsilon}_0$,

$$\boldsymbol{\epsilon}_0 = (1+\nu)[\alpha\Delta T, \alpha\Delta T, 0]^{\mathrm{T}} \tag{1.23}$$

For plane stress and plane strain, note that $\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \tau_{xy}]^{\mathrm{T}}$ and $\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \gamma_{xy}]^{\mathrm{T}}$, and that **D** matrices are as given in Eqs. 1.18 and 1.19, respectively.

1.9 POTENTIAL ENERGY AND EQUILIBRIUM; THE RAYLEIGH–RITZ METHOD

In mechanics of solids, our problem is to determine the displacement $\mathbf{u}$ of the body shown in Fig. 1.1, satisfying the equilibrium equations 1.6. Note that stresses are related to strains, which, in turn, are related to displacements. This leads to requiring solution of second-order partial differential equations. Solution of this set of equations is generally referred to as an *exact* solution. Such exact solutions are available for simple geometries and loading conditions, and one may refer to publications in theory of elasticity. For problems of complex geometries and general boundary and loading conditions, obtaining such solutions is an almost impossible task. Approximate solution methods usually employ potential energy or variational methods, which place less stringent conditions on the functions.

Potential Energy, Π

The total potential energy Π of an elastic body, is defined as the sum of total strain energy (U) and the work potential:

$$\Pi = \underset{(U)}{\text{Strain energy}} + \underset{(\mathrm{WP})}{\text{Work potential}} \tag{1.24}$$

For linear elastic materials, the strain energy per unit volume in the body is $\frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}$. For the elastic body shown in Fig. 1.1, the total strain energy U is given by

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dV \tag{1.25}$$

The work potential WP is given by

$$\mathrm{WP} = -\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\, dV - \int_S \mathbf{u}^{\mathrm{T}}\mathbf{T}\, dS - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{1.26}$$

The total potential for the general elastic body shown in Fig. 1.1 is

$$\Pi = \frac{1}{2}\int_V \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dV - \int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\, dV - \int_S \mathbf{u}^{\mathrm{T}}\mathbf{T}\, dS - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{1.27}$$

We consider conservative systems here, where the work potential is independent of the path taken. In other words, if the system is displaced from a given configuration and brought back to this state, the forces do zero work regardless of the path. The potential energy principle is now stated as follows:

> **Principle of Minimum Potential Energy**
>
> For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

4

#

Example 1.2

The potential energy for the linear elastic one-dimensional rod (Fig. E1.2), with body force neglected, is

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - 2u_1$$

where $u_1 = u(x = 1)$.

Let us consider a polynomial function

$$u = a_1 + a_2x + a_3x^2$$

This must satisfy $u = 0$ at $x = 0$ and $u = 0$ at $x = 2$. Thus,

$$0 = a_1$$
$$0 = a_1 + 2a_2 + 4a_3$$

Hence,

$$a_2 = -2a_3$$
$$u = a_3(-2x + x^2) \qquad u_1 = -a_3$$

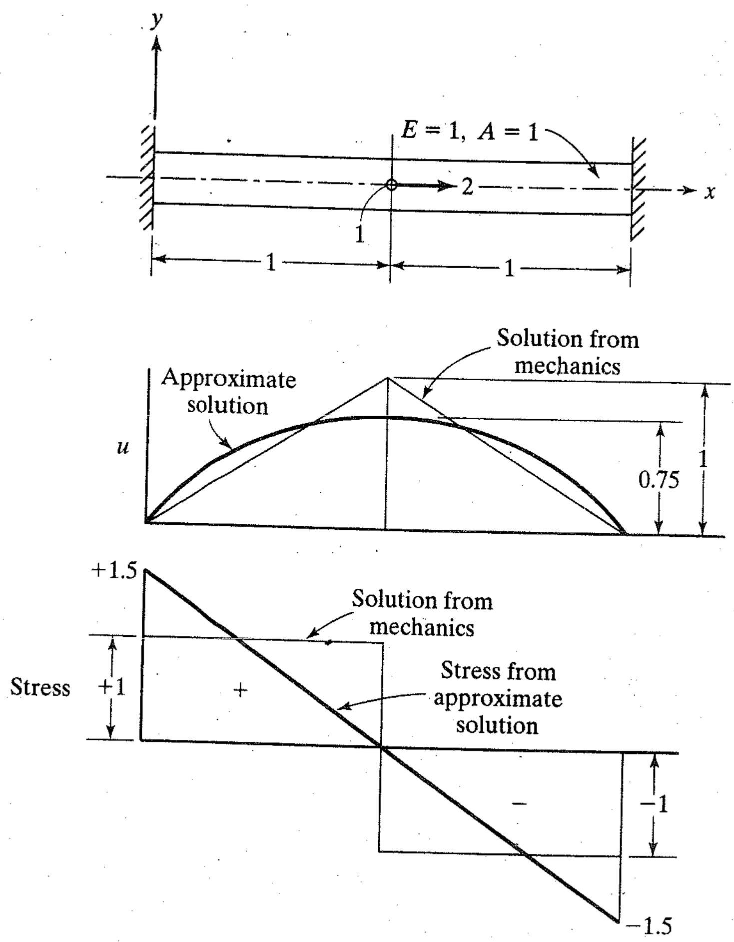


FIGURE E1.2

Then $du/dx = 2a_3(-1 + x)$ and

$$\Pi = \frac{1}{2}\int_0^2 4a_3^2(-1 + x)^2\,dx - 2(-a_3)$$

$$= 2a_3^2\int_0^2 (1 - 2x + x^2)\,dx + 2a_3$$

$$= 2a_3^2\left(\tfrac{2}{3}\right) + 2a_3$$

We set $\partial\Pi/\partial a_3 = 4a_3\left(\frac{2}{3}\right) + 2 = 0$, resulting in

$$a_3 = -0.75 \qquad u_1 = -a_3 = 0.75$$

The stress in the bar is given by

$$\sigma = E\frac{du}{dx} = 1.5(1 - x)$$

■

We note here that an exact solution is obtained if piecewise polynomial interpolation is used in the construction of u.

The finite element method provides a systematic way of constructing the basis functions ϕ_i used in Eq. 1.30.

1.10 GALERKIN'S METHOD

Galerkin's method uses the set of governing equations in the development of an integral form. It is usually presented as one of the weighted residual methods. For our discussion, let us consider a general representation of a governing equation on a region V:

$$Lu = P \tag{1.33}$$

For the one-dimensional rod considered in Example 1.2, the governing equation is the differential equation

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

We may consider L as the operator

$$\frac{d}{dx}EA\frac{d}{dx}(\,)$$

operating on u.

The exact solution needs to satisfy (1.33) at every point x. If we seek an approximate solution $\tilde{u}$, it introduces an error $\epsilon(x)$, called the *residual*:

$$\epsilon(x) = L\tilde{u} - P \tag{1.34}$$

The approximate methods revolve around setting the residual relative to a weighting function W_i, to zero:

$$\int_V W_i(L\tilde{u} - P)\,dV = 0 \qquad i = 1 \text{ to } n \tag{1.35}$$

Summary: Elimination Approach

Consider the boundary conditions

$$Q_{p_1} = a_1, Q_{p_2} = a_2, \ldots, Q_{p_r} = a_r$$

Step 1. Store the p_1th, p_2th, ..., and p_rth rows of the global stiffness matrix **K** and force vector **F**. These rows will be used subsequently.

Step 2. Delete the p_1th row and column, the p_2th row and column, ..., and the p_rth row and column from the **K** matrix. The resulting stiffness matrix **K** is of dimension $(N - r, N - r)$. Similarly, the corresponding load vector **F** is of dimension $(N - r, 1)$. Modify each load component as

$$F_i = F_i - (K_{i,p_1}a_1 + K_{i,p_2}a_2 + \cdots + K_{i,p_r}a_r) \tag{3.70}$$

for each dof i that is not a support. Solve

$$\mathbf{KQ} = \mathbf{F}$$

for the displacement vector **Q**.

Step 3. For each element, extract the element displacement vector **q** from the **Q** vector, using element connectivity, and determine element stresses.

Step 4. Using the information stored in step 1, evaluate the reaction forces at each support dof from

$$\begin{aligned} R_{p_1} &= K_{p_1 1}Q_1 + K_{p_1 2}Q_2 + \cdots + K_{p_1 N}Q_N - F_{p_1} \\ R_{p_2} &= K_{p_2 1}Q_1 + K_{p_2 2}Q_2 + \cdots + K_{p_2 N}Q_N - F_{p_2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ R_{p_r} &= K_{p_r 1}Q_1 + K_{p_r 2}Q_2 + \cdots + K_{p_r N}Q_N - F_{p_r} \end{aligned} \tag{3.71}$$

Example 3.3

Consider the thin (steel) plate in Fig. E3.3a. The plate has a uniform thickness $t = 1$ in., Young's modulus $E = 30 \times 10^6$ psi, and weight density $\rho = 0.2836$ lb/in.^{3}. In addition to its self-weight, the plate is subjected to a point load $P = 100$ lb at its midpoint.

(a) Model the plate with two finite elements.

(b) Write down expressions for the element stiffness matrices and element body force vectors.

(c) Assemble the structural stiffness matrix **K** and global load vector **F**.

(d) Using the elimination approach, solve for the global displacement vector **Q**.

(e) Evaluate the stresses in each element.

(f) Determine the reaction force at the support.

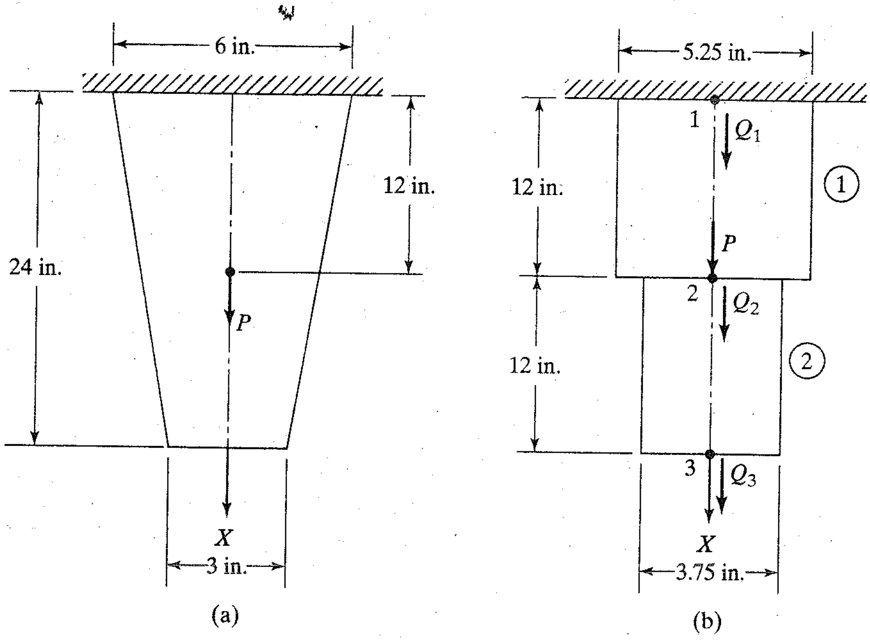


FIGURE E3.3

Solution

(a) Using two elements, each of 12 in. in length, we obtain the finite element model in Fig. E3.3b. Nodes and elements are numbered as shown. Note that the area at the midpoint of the plate in Fig. E3.3a is 4.5 in.^{2}. Consequently, the average area of element 1 is $A_1 = (6 + 4.5)/2 = 5.25$ in.^{2}, and the average area of element 2 is $A_2 = (4.5 + 3)/2 = 3.75$ in.^{2}. The boundary condition for this model is $Q_1 = 0$.

(b) From Eq. 3.26, we can write down expressions for the element stiffness matrices of the two elements as

$$\mathbf{k}^1 = \frac{30 \times 10^6 \times 5.25}{12} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \leftarrow \text{Global dof} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

$$K = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$\mathbf{k}^2 = \frac{30 \times 10^6 \times 3.75}{12} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

Using Eq. 3.31, the element body force vectors are

$$\mathbf{f}^1 = \frac{5.25 \times 12 \times 0.2836}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \begin{array}{c} \text{Global dof} \\ \downarrow \\ 1 \\ 2 \end{array}$$

$$f = \frac{A_e l_e f}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

and

$$\mathbf{f}^2 = \frac{3.75 \times 12 \times 0.2836}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \begin{array}{c} 2 \\ 3 \end{array}$$

$\boxed{K \cdot Q = F}$

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(c) The global stiffness matrix $\mathbf{K}$ is assembled from $\mathbf{k}^1$ and $\mathbf{k}^2$ as

$$\mathbf{K} = \frac{30 \times 10^6}{12} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5.25 & -5.25 & 0 \\ -5.25 & 9.00 & -3.75 \\ 0 & -3.75 & 3.75 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

The externally applied global load vector $\mathbf{F}$ is assembled from $\mathbf{f}^1$, $\mathbf{f}^2$, and the point load $P = 100$ lb; as

$$\mathbf{F} = \begin{Bmatrix} 8.9334 \\ 15.3144 + 100 \\ 6.3810 \end{Bmatrix}$$

(d) In the elimination approach, the stiffness matrix $\mathbf{K}$ is obtained by deleting rows and columns corresponding to fixed dofs. In this problem, dof 1 is fixed. Thus, $\mathbf{K}$ is obtained by deleting the first row and column of the original $\mathbf{K}$. Also, $\mathbf{F}$ is obtained by deleting the first component of the original $\mathbf{F}$. The resulting equations are

$$\frac{30 \times 10^6}{12} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 9.00 & -3.75 \\ -3.75 & 3.75 \end{bmatrix} \end{array} \begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 115.3144 \\ 6.3810 \end{Bmatrix}$$

Solution of these equations yields

$$Q_2 = 0.9272 \times 10^{-5} \text{ in.}$$
$$Q_3 = 0.9953 \times 10^{-5} \text{ in.}$$

Thus, $Q = [0, 0.9272 \times 10^{-5}, 0.9953 \times 10^{-5}]^T$ in.

(e) Using Eqs. 3.15 and 3.16, we obtain the stress in each element:

$$\sigma_1 = 30 \times 10^6 \times \tfrac{1}{12}[-1 \quad 1]\begin{Bmatrix} 0 \\ 0.9272 \times 10^{-5} \end{Bmatrix}$$
$$= 23.18 \text{ psi}$$

and

$$\sigma_2 = 30 \times 10^6 \times \tfrac{1}{12}[-1 \quad 1]\begin{Bmatrix} 0.9272 \times 10^{-5} \\ 0.9953 \times 10^{-5} \end{Bmatrix}$$
$$= 1.70 \text{ psi}$$

(f) The reaction force R_1 at node 1 is obtained from Eq. 3.71. This calculation requir the first row of $\mathbf{K}$ from part (c). Also, from part (c), note that the externally applie load (due to the self-weight) at note 1 is $F_1 = 8.9334$ lb. Thus,

$$R_1 = \frac{30 \times 10^6}{12}[5.25 \quad -5.25 \quad 0]\begin{Bmatrix} 0 \\ 0.9272 \times 10^{-5} \\ 0.9953 \times 10^{-5} \end{Bmatrix} - 8.9334$$
$$= -130.6 \text{ lb}$$

Evidently, the reaction is equal and opposite to the total downward load on t plate.

(7)

It should be noted that the penalty approach presented herein is an approximate approach. The accuracy of the solution, particularly the reaction forces, depends on the choice of C.

Choice of C. Let us expand the first equation in Eq. 3.74. We have

$$(K_{11} + C)Q_1 + K_{12}Q_2 + \cdots + K_{1N}Q_N = F_1 + Ca_1 \tag{3.79a}$$

Upon dividing by C, we get

$$\left(\frac{K_{11}}{C} + 1\right)Q_1 + \frac{K_{12}}{C}Q_2 + \cdots + \frac{K_{1N}}{C}Q_N = \frac{F_1}{C} + a_1 \tag{3.79b}$$

From this equation, we see that if C is chosen large enough, then $Q_1 \approx a_1$. Specifically, we see that if C is large compared to the stiffness coefficients $K_{11}, K_{12}, \ldots, K_{1N}$, then $Q_1 \approx a_1$. Note that F_1 is a load applied at the support (if any), and that F_1/C is generally of small magnitude.

A simple scheme suggests itself for choosing the magnitude of C:

$$C = \max|K_{ij}| \times 10^4$$

for

$$1 \le i \le N \qquad 1 \le j \le N \tag{3.80}$$

The choice of 10^4 has been found to be satisfactory on most computers. The reader may wish to choose a sample problem and experiment with this (using, say, 10^5 or 10^6) to check whether the reaction forces differ by much.

#

Example 3.4

Consider the bar shown in Fig. E3.4. An axial load $P = 200 \times 10^3$ N is applied as shown. Using the penalty approach for handling boundary conditions, do the following:

(a) Determine the nodal displacements.
(b) Determine the stress in each material.
(c) Determine the reaction forces.

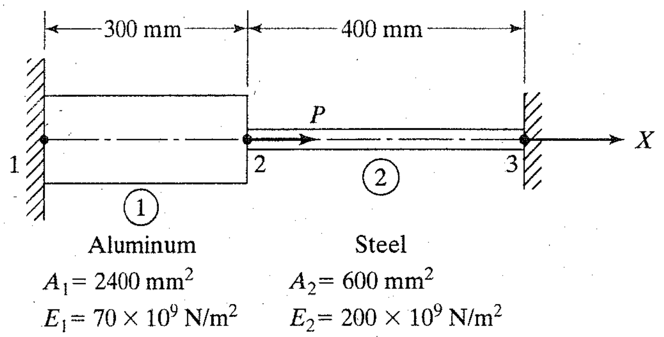

FIGURE E3.4

Solution

(a) The element stiffness matrices are

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \leftarrow \text{Global dof} \\ \mathbf{k}^1 = \dfrac{70 \times 10^3 \times 2400}{300} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

and

$$\begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ \mathbf{k}^2 = \dfrac{200 \times 10^3 \times 600}{400} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

The structural stiffness matrix that is assembled from $\mathbf{k}^1$ and $\mathbf{k}^2$ is

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \mathbf{K} = 10^6 & \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 0.30 \end{bmatrix} \end{array}$$

The global load vector is

$$\mathbf{F} = [0, \quad 200 \times 10^3, \quad 0]^{\mathrm{T}}$$

Now dofs 1 and 3 are fixed. When using the penalty approach, therefore, a large number C is added to the first and third diagonal elements of $\mathbf{K}$. Choosing C based on Eq. 3.80, we get

$$C = [0.86 \times 10^6] \times 10^4$$

Thus, the modified stiffness matrix is

$$\mathbf{K} = 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix}$$

The finite element equations are given by

$$10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{Bmatrix}$$

which yields the solution

$$\mathbf{Q} = [15.1432 \times 10^{-6}, \quad 0.23257, \quad 8.1127 \times 10^{-6}]^{\mathrm{T}}\ \text{mm}$$

(b) The element stresses (Eq. 3.16) are

$$\sigma_1 = 70 \times 10^3 \times \frac{1}{300}[-1 \quad 1] \begin{Bmatrix} 15.1432 \times 10^{-6} \\ 0.23257 \end{Bmatrix}$$
$$= 54.27\ \text{MPa}$$

where 1 MPa $= 10^6\,\text{N/m}^2 = 1\,\text{N/mm}^2$. Also,

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{400}[-1 \quad 1]\begin{Bmatrix} 0.23257 \\ 8.1127 \times 10^{-6} \end{Bmatrix}$$
$$= -116.29\,\text{MPa}$$

(c) The reaction forces are obtained from Eq. 3.78 as

$$\begin{aligned} R_1 &= -CQ_1 \\ &= -[0.86 \times 10^{10}] \times 15.1432 \times 10^{-6} \\ &= -130.23 \times 10^3 \end{aligned}$$

Also,

$$\begin{aligned} R_3 &= -CQ_3 \\ &= -[0.86 \times 10^{10}] \times 8.1127 \times 10^{-6} \\ &= -69.77 \times 10^3\,\text{N} \end{aligned}$$

■

Example 3.5

In Fig. E3.5a, a load $P = 60 \times 10^3$ N is applied as shown. Determine the displacement field, stress, and support reactions in the body. Take $E = 20 \times 10^3\,\text{N/mm}^2$.

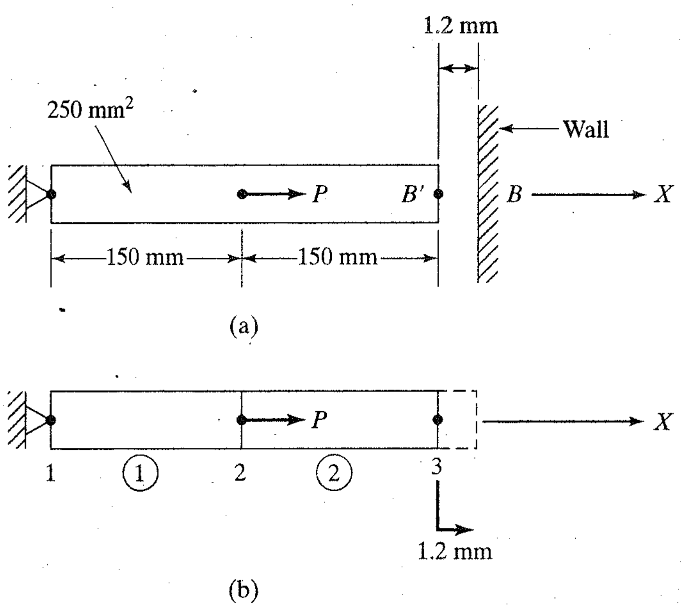

FIGURE E3.5

Solution In this problem, we should first determine whether contact occurs between the bar and the wall, B. To do this, assume that the wall does not exist. Then, the solution to the problem can be verified to be

$$Q_{B'} = 1.8\,\text{mm}$$

where $Q_{B'}$ is the displacement of point B'. From this result, we see that contact does occur. The problem has to be re-solved, since the boundary conditions are now different: The displacement

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The derivation of the result $\mathbf{k} = \mathbf{L}^{\mathrm{T}}\mathbf{k}'\mathbf{L}$ also follows from Galerkin's variational principle. The virtual work δW as a result of virtual displacement $\boldsymbol{\psi}'$ is

$$\delta W = \boldsymbol{\psi}'^{\mathrm{T}}(\mathbf{k}'\mathbf{q}') \tag{4.14a}$$

Since $\boldsymbol{\psi}' = \mathbf{L}\boldsymbol{\psi}$ and $\mathbf{q}' = \mathbf{L}\mathbf{q}$, we have

$$\begin{aligned} \delta W &= \boldsymbol{\psi}^{\mathrm{T}}[\mathbf{L}^{\mathrm{T}}\mathbf{k}'\mathbf{L}]\mathbf{q} \\ &= \boldsymbol{\psi}^{\mathrm{T}}\mathbf{k}\mathbf{q} \end{aligned} \tag{4.14b}$$

Stress Calculations

Expressions for the element stresses can be obtained by noting that a truss element in local coordinates is a simple two-force member (Fig. 4.2). Thus, the stress σ in a truss element is given by

$$\sigma = E_e \epsilon \tag{4.15a}$$

Since the strain ϵ is the change in length per unit original length,

$$\begin{aligned} \sigma &= E_e \frac{q_2' - q_1'}{\ell_e} \\ &= \frac{E_e}{\ell_e}[-1 \quad 1]\begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix} \end{aligned} \tag{4.15b}$$

This equation can be written in terms of the global displacements $\mathbf{q}$ using the transformation $\mathbf{q}' = \mathbf{L}\mathbf{q}$ as

$$\sigma = \frac{E_e}{\ell_e}[-1 \quad 1]\mathbf{L}\mathbf{q} \tag{4.15c}$$

Substituting for $\mathbf{L}$ from Eq. 4.5 yields

$$\sigma = \frac{E_e}{\ell_e}[-\ell \quad -m \quad \ell \quad m]\mathbf{q} \tag{4.16}$$

Once the displacements are determined by solving the finite element equations, the stresses can be recovered from Eq. 4.16 for each element. Note that a positive stress implies that the element is in tension and a negative stress implies compression.

#

Example 4.1

Consider the four-bar truss shown in Fig. E4.1a. It is given that $E = 29.5 \times 10^6$ psi and $A_e = 1$ in.^{2} for all elements. Complete the following:

(a) Determine the element stiffness matrix for each element.
(b) Assemble the structural stiffness matrix $\mathbf{K}$ for the entire truss.
(c) Using the elimination approach, solve for the nodal displacement.
(d) Recover the stresses in each element.
(e) Calculate the reaction forces.

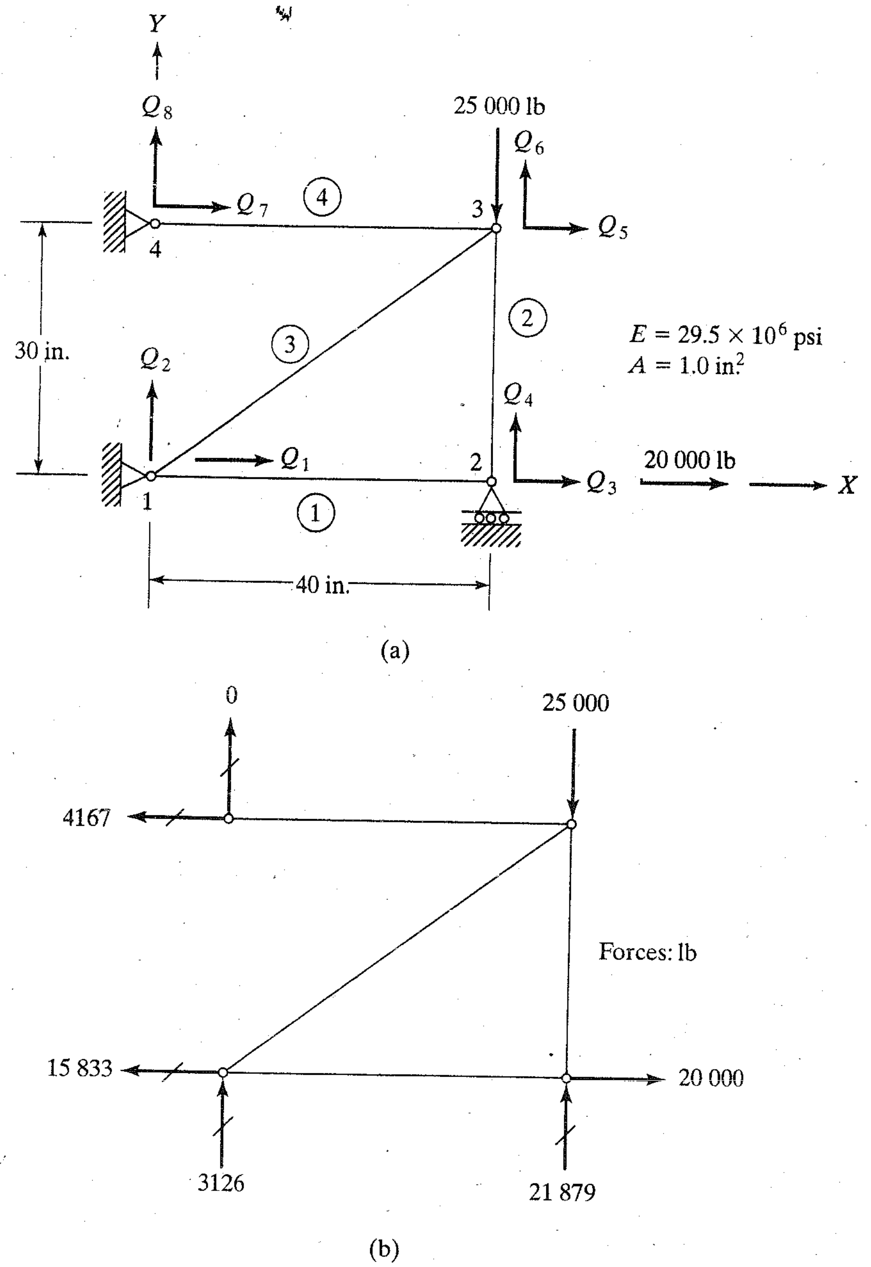

FIGURE E4.1

Solution

(a) It is recommended that a *tabular* form be used for representing nodal coordinate data and element information. The nodal coordinate data are as follows:

Node	x	y
1	0	0
2	40	0
3	40	30
4	0	30

The element connectivity table is

Element	1	2
1	1	2
2	3	2
3	1	3
4	4	3

Note that the user has a choice in defining element connectivity. For example, the connectivity of element 2 can be defined as 2–3 instead of 3–2 as in the previous table. However, calculations of the direction cosines will be consistent with the adopted connectivity scheme. Using formulas in Eqs. 4.6 and 4.7, together with the nodal coordinate data and the given element connectivity information, we obtain the direction cosines table:

Element	ℓ_e	ℓ	m
1	40	1	0
2	30	0	−1
3	50	0.8	0.6
4	40	1	0

For example, the direction cosines of elements 3 are obtained as $\ell = (x_3 - x_1)/\ell_e = (40 - 0)/50 = 0.8$ and $m = (y_3 - y_1)/\ell_e = (30 - 0)/50 = 0.6$. Now, using Eq. 4.13, the element stiffness matrices for element 1 can be written as

$$\mathbf{k}^1 = \frac{29.5 \times 10^6}{40} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \leftarrow \text{Global dof} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

The global dofs associated with element 1, which is connected between nodes 1 and 2, are indicated in $\mathbf{k}^1$ earlier. These global dofs are shown in Fig. E4.1a and assist in assembling the various element stiffness matrices.

The element stiffness matrices of elements 2, 3, and 4 are as follows:

$$\mathbf{k}^2 = \frac{29.5 \times 10^6}{30} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 3 \\ 4 \end{array}$$

$$\mathbf{k}^3 = \frac{29.5 \times 10^6}{50} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} .64 & .48 & -.64 & -.48 \\ .48 & .36 & -.48 & -.36 \\ -.64 & -.48 & .64 & .48 \\ -.48 & -.36 & .48 & .36 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$\mathbf{k}^4 = \frac{29.5 \times 10^6}{40} \begin{array}{c} \begin{array}{cccc} 7 & 8 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 7 \\ 8 \\ 5 \\ 6 \end{array}$$

(b) The structural stiffness matrix **K** is now assembled from the element stiffness matrices. By adding the element stiffness contributions, noting the element connectivity, we get

$$\mathbf{K} = \frac{29.5 \times 10^6}{600} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ -15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ -7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15.0 & 0 \\ -5.76 & -4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

(c) The structural stiffness matrix **K** given above needs to be modified to account for the boundary conditions. The elimination approach discussed in Chapter 3 will be used here. The rows and columns corresponding to dofs 1, 2, 4, 7, and 8, which correspond to fixed supports, are deleted from the **K** matrix. The reduced finite element equations are given as

$$\frac{29.5 \times 10^6}{600} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix} \begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 20\,000 \\ 0 \\ -25\,000 \end{Bmatrix}$$

Solution of these equations yields the displacements

$$\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 27.12 \times 10^{-3} \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \end{Bmatrix} \text{in.}$$

The nodal displacement vector for the entire structure can therefore be written as

$$\mathbf{Q} = [0, 0, 27.12 \times 10^{-3}, 0, 5.65 \times 10^{-3}, -22.25 \times 10^{-3}, 0, 0]^T \text{ in.}$$

(d) The stress in each element can now be determined from Eq. 4.16, as shown below.

The connectivity of element 1 is 1 – 2. Consequently, the nodal displacement vector for element 1 is given by $\mathbf{q} = [0, 0, 27.12 \times 10^{-3}, 0]^T$, and Eq. 4.16 yields

$$\sigma_1 = \frac{29.5 \times 10^6}{40}[-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$= 20\,000.0 \text{ psi}$$

The stress in member 2 is given by

$$\sigma_2 = \frac{29.5 \times 10^6}{30}[0 \quad 1 \quad 0 \quad -1] \begin{Bmatrix} 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ -27.12 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$= -21\,880.0 \text{ psi}$$

Following similar steps, we get

$$\sigma_3 = 5208.0 \text{ psi}$$
$$\sigma_4 = 4167.0 \text{ psi}$$

(e) The final step is to determine the support reactions. We need to determine the reaction forces along dofs 1, 2, 4, 7, and 8, which correspond to fixed supports. These are obtained by substituting for **Q** into the original finite element equation **R** = **KQ** − **F**. In this substitution, only those rows of **K** corresponding to the support dofs are needed, and **F** = **O** for these dofs. Thus, we have

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \frac{29.5 \times 10^6}{600} \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

which results in

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} -15833.0 \\ 3126.0 \\ 21879.0 \\ -4167.0 \\ 0 \end{Bmatrix} \text{lb}$$

A free body diagram of the truss with reaction forces and applied loads is shown in Fig. E4.1b. ■

Temperature Effects

The thermal stress problem is considered here. Since a truss element is simply a one-dimensional element when viewed in the local coordinate system, the element temperature load in the local coordinate system is given by (see Eq. 3.103b)

$$\mathbf{\Theta}' = E_e A_e \epsilon_0 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \tag{4.17}$$

where the initial strain ϵ_0 associated with a temperature change is given by

$$\epsilon_0 = \alpha\,\Delta T \tag{4.18}$$

in which α is the coefficient of thermal expansion, and ΔT is the average change in temperature in the element. It may be noted that the initial strain ϵ_0 can also be induced by forcing members into places that are either too long or too short, due to fabrication errors.

We will now express the load vector in Eq. 4.17 in the global coordinate system. Since the potential energy associated with this load is the same in magnitude whether measured in the local or global coordinate systems, we have

$$\mathbf{q}'^{\mathrm{T}}\mathbf{\Theta}' = \mathbf{q}^{\mathrm{T}}\mathbf{\Theta} \tag{4.19}$$

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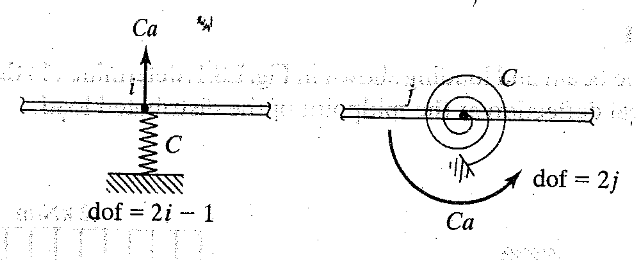

a = known generalized displacement

FIGURE 8.7 Boundary conditions for a beam.

8.5 SHEAR FORCE AND BENDING MOMENT

Using the bending moment and shear force equations

$$M = EI\frac{d^2v}{dx^2} \qquad V = \frac{dM}{dx} \quad \text{and} \quad v = \mathbf{Hq}$$

we get the element bending moment and shear force:

$$M = \frac{EI}{\ell_e^2}[6\xi q_1 + (3\xi - 1)\ell_e q_2 - 6\xi q_3 + (3\xi + 1)\ell_e q_4] \tag{8.38}$$

$$V = \frac{6EI}{\ell_e^3}(2q_1 + \ell_e q_2 - 2q_3 + \ell_e q_4) \tag{8.39}$$

These bending moment and shear force values are for the loading as modeled using equivalent point loads. Denoting element end equilibrium loads as R_1, R_2, R_3, and R_4, we note that

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \frac{EI}{\ell_e^3}\begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ 6\ell_e & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ -12 & -6\ell_e & 12 & -6\ell_e \\ 6\ell_e & 2\ell_e^2 & -6\ell_e & 4\ell_e^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} + \begin{Bmatrix} \dfrac{-p\ell_e}{2} \\ \dfrac{-p\ell_e^2}{12} \\ \dfrac{-p\ell_e}{2} \\ \dfrac{p\ell_e^2}{12} \end{Bmatrix} \tag{8.40}$$

It is easily seen that the first term on the right is $\mathbf{k}^e\mathbf{q}$. Also note that the second term needs to be added only on elements with distributed load. In books on matrix structural analysis, the previous equations are written directly from element equilibrium. Also, the last vector on the right side of the equation consists of terms that are called *fixed-end reactions*. The shear forces at the two ends of the element are $V_1 = R_1$ and $V_2 = -R_3$. The end bending moments are $M_1 = -R_2$ and $M_2 = R_4$.

Example 8.1

For the beam and loading shown in Fig. E8.1, determine (1) the slopes at 2 and 3 and (2) the vertical deflection at the midpoint of the distributed load.

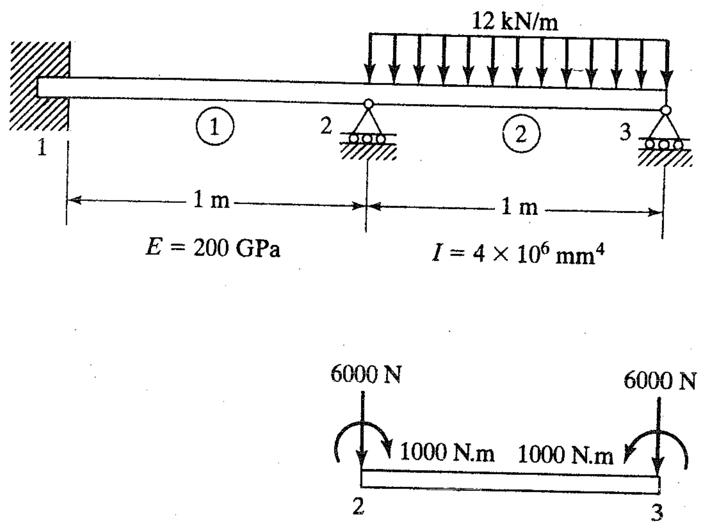

FIGURE E8.1

Solution We consider the two elements formed by the three nodes. Displacements Q_1, Q_2, Q_3, and Q_5 are constrained to be zero, and Q_4 and Q_6 need to be found. Since the lengths and sections are equal, the element matrices are calculated from Eq. 8.29 as follows:

$$\frac{EI}{\ell^3} = \frac{(200 \times 10^9)(4 \times 10^{-6})}{1^3} = 8 \times 10^5\ \text{N/m}$$

$$\mathbf{k}^1 = \mathbf{k}^2 = 8 \times 10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$\begin{array}{lcccc} e = 1 & Q_1 & Q_2 & Q_3 & Q_4 \\ e = 2 & Q_3 & Q_4 & Q_5 & Q_6 \end{array}$$

We note that global applied loads are $F_4 = -1000$ N.m and $F_6 = +1000$ N.m obtained from $p\ell^2/12$, as seen in Fig. 8.6. We use here the elimination approach presented in Chapter 3. Using the connectivity, we obtain the global stiffness after elimination:

$$\mathbf{K} = \begin{bmatrix} k_{44}^{(1)} + k_{22}^{(2)} & k_{24}^{(2)} \\ k_{42}^{(2)} & k_{44}^{(2)} \end{bmatrix}$$

$$= 8 \times 10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix}$$

The set of equations is given by

$$8 \times 10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ +1000 \end{Bmatrix}$$

The solution is

$$\begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -2.679 \times 10^{-4} \\ 4.464 \times 10^{-4} \end{Bmatrix}$$

For element 2, $q_1 = 0, q_2 = Q_4, q_3 = 0$, and $q_4 = Q_6$. To get vertical deflection at the midpoint of the element, use $v = \mathbf{Hq}$ at $\xi = 0$:

$$\begin{aligned} v &= 0 + \frac{\ell_e}{2} H_2 Q_4 + 0 + \frac{\ell_e}{2} H_4 Q_6 \\ &= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)(-2.679 \times 10^{-4}) + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{4}\right)(4.464 \times 10^{-4}) \\ &= -8.93 \times 10^{-5}\,\text{m} \\ &= -0.0893\,\text{mm} \end{aligned}$$

■

8.6 BEAMS ON ELASTIC SUPPORTS

In many engineering applications, beams are supported on elastic members. Shafts are supported on ball, roller, or journal bearings. Large beams are supported on elastic walls. Beams supported on soil form a class of applications known as Winkler foundations.

Single-row ball bearings can be considered by having a node at each bearing location and adding the bearing stiffness k_B to the diagonal location of vertical degree of freedom (Fig. 8.8a). Rotational (moment) stiffness has to be considered for roller bearings and journal bearings.

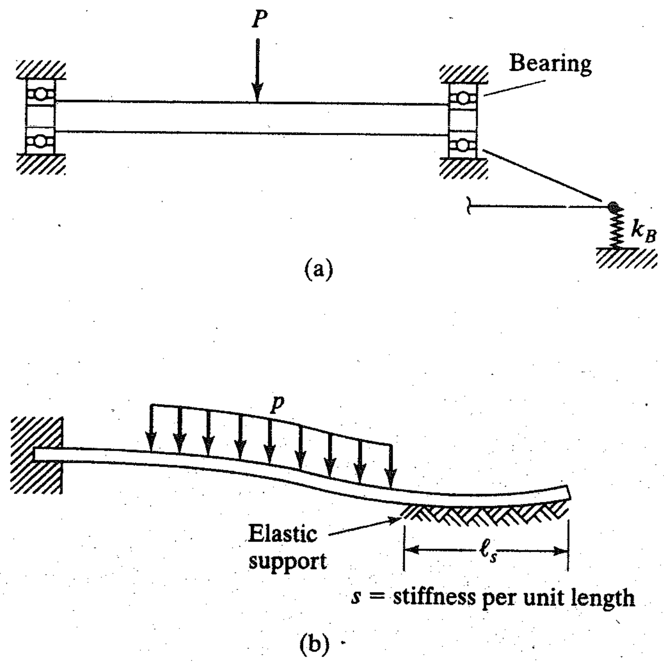

FIGURE 8.8 Elastic support.

In wide journal bearings and Winkler foundations, we use stiffness per unit length, s, of the supporting medium (Fig. 8.8b). Over the length of the support, this adds the following term to the total potential energy:

$$\frac{1}{2}\int_0^\ell sv^2\,dx \tag{8.41}$$

In Galerkin's approach, this term is $\int_0^\ell sv\phi\,dx$. When we substitute for $v = \mathbf{Hq}$ for the discretized model, the previous term becomes

$$\frac{1}{2}\sum_e \mathbf{q}^{\mathrm{T}} s \int_e \mathbf{H}^{\mathrm{T}}\mathbf{H}\,dx\,\mathbf{q} \tag{8.42}$$

We recognize the stiffness term in this summation, namely,

$$\mathbf{k}_s^e = s\int_e \mathbf{H}^{\mathrm{T}}\mathbf{H}\,dx = \frac{s\ell_e}{2}\int_{-1}^{+1}\mathbf{H}^{\mathrm{T}}\mathbf{H}\,d\xi \tag{8.43}$$

On integration, we have

$$\mathbf{k}_s^e = \frac{s\ell_e}{420}\begin{bmatrix} 156 & 22\ell_e & 54 & -13\ell_e \\ 22\ell_e & 4\ell_e^2 & 13\ell_e & -3\ell_e^2 \\ 54 & 13\ell_e & 156 & -22\ell_e \\ -13\ell_e & -3\ell_e^2 & -22\ell_e & 4\ell_e^2 \end{bmatrix} \tag{8.44}$$

For elements supported on an elastic foundation, this stiffness has to be added to the element stiffness given by Eq. 8.29. Matrix $\mathbf{k}_s^e$ is the consistent stiffness matrix for the elastic foundation.

8.7 PLANE FRAMES

Here, we consider plane structures with rigidly connected members. These members will be similar to the beams except that axial loads and axial deformations are present. The elements also have different orientations. Figure 8.9 shows a frame element. We have two displacements and a rotational deformation for each node. The nodal displacement vector is given by

$$\mathbf{q} = [q_1, q_2, q_3, q_4, q_5, q_6]^{\mathrm{T}} \tag{8.45}$$

We also define the local or body coordinate system x', y', such that x' is oriented along 1–2, with direction cosines ℓ, m (where $\ell = \cos\theta$, $m = \sin\theta$). These are evaluated using relationships given for the truss element, shown in Fig. 4.4. The nodal displacement vector in the local system is

$$\mathbf{q}' = [q_1', q_2', q_3', q_4', q_5', q_6']^{\mathrm{T}} \tag{8.46}$$

Recognizing that $q_3' = q_3$ and $q_6' = q_6$, which are rotations with respect to the body, we obtain the local–global transformation

$$\mathbf{q}' = \mathbf{Lq} \tag{8.47}$$

where

(14)

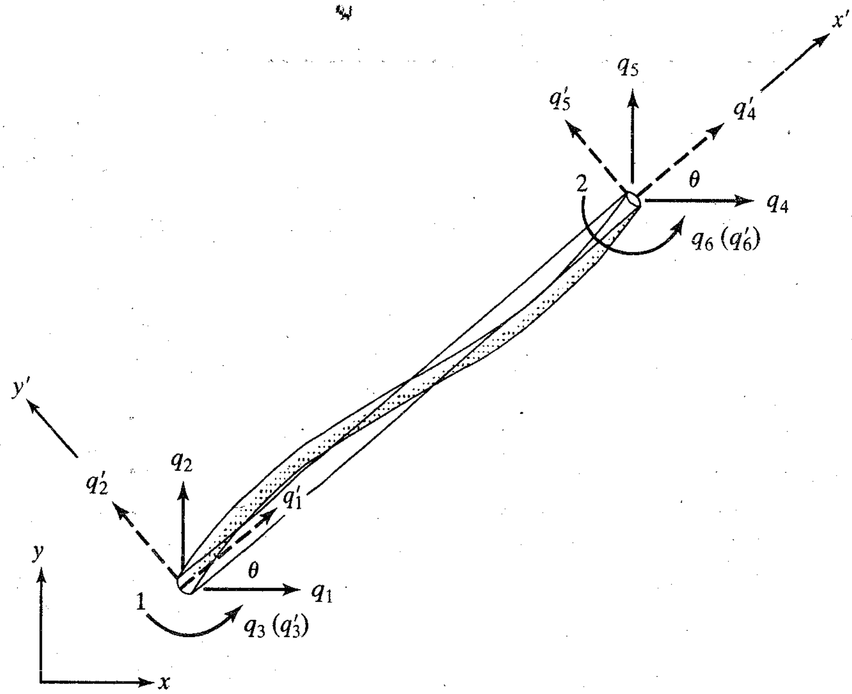

FIGURE 8.9 Frame element.

$$\mathbf{L} = \begin{bmatrix} \ell & m & 0 & 0 & 0 & 0 \\ -m & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ell & m & 0 \\ 0 & 0 & 0 & -m & \ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8.48}$$

It is now observed that q'_2, q'_3, q'_5, and q'_6, are like the beam degrees of freedom, while q'_1 and q'_4 are similar to the displacements of a rod element, as discussed in Chapter 3. Combining the two stiffnesses and arranging in proper locations, we get the element stiffness for a frame element as

$$\mathbf{k}'^e = \begin{bmatrix} \frac{EA}{\ell_e} & 0 & 0 & \frac{-EA}{\ell_e} & 0 & 0 \\ 0 & \frac{12EI}{\ell_e^3} & \frac{6EI}{\ell_e^2} & 0 & \frac{-12EI}{\ell_e^3} & \frac{6EI}{\ell_e^2} \\ 0 & \frac{6EI}{\ell_e^2} & \frac{4EI}{\ell_e} & 0 & \frac{-6EI}{\ell_e^2} & \frac{2EI}{\ell_e} \\ \frac{-EA}{\ell_e} & 0 & 0 & \frac{EA}{\ell_e} & 0 & 0 \\ 0 & \frac{-12EI}{\ell_e^3} & \frac{-6EI}{\ell_e^2} & 0 & \frac{12EI}{\ell_e^3} & \frac{-6EI}{\ell_e^2} \\ 0 & \frac{6EI}{\ell_e^2} & \frac{2EI}{\ell_e} & 0 & \frac{-6EI}{\ell_e^2} & \frac{4EI}{\ell_e} \end{bmatrix} \tag{8.49}$$

As discussed in the development of a truss element in Chapter 4, we recognize that the element strain energy is given by

$$U_e = \tfrac{1}{2}\mathbf{q}'^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{q}' = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{L}\mathbf{q} \tag{8.50}$$

or in Galerkin's approach, the internal virtual work of an element is

$$W_e = \boldsymbol{\psi}'^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{q}' = \boldsymbol{\psi}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{L}\mathbf{q} \tag{8.51}$$

where $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}$ are virtual nodal displacements in local and global coordinate systems, respectively. From Eq. 8.50 or 8.51, we recognize the element stiffness matrix in global coordinates to be

$$\boxed{\mathbf{k}^{e} = \mathbf{L}^{\mathrm{T}}\mathbf{k}'^{e}\mathbf{L}} \tag{8.52}$$

In the finite element program implementation, $\mathbf{k}'^{e}$ can first be defined, and then this matrix multiplication can be carried out.

If there is distributed load on a member, as shown in Fig. 8.10, we have

$$\mathbf{q}'^{\mathrm{T}}\mathbf{f}' = \mathbf{q}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{f}' \tag{8.53}$$

where

$$\mathbf{f}' = \left[0, \ \frac{p\ell_e}{2}, \ \frac{p\ell_e^2}{12}, \ 0, \ \frac{p\ell_e}{2}, \ -\frac{p\ell_e^2}{12}\right]^{\mathrm{T}} \tag{8.54}$$

The nodal loads due to the distributed load p are given by

$$\mathbf{f} = \mathbf{L}^{\mathrm{T}}\mathbf{f}' \tag{8.55}$$

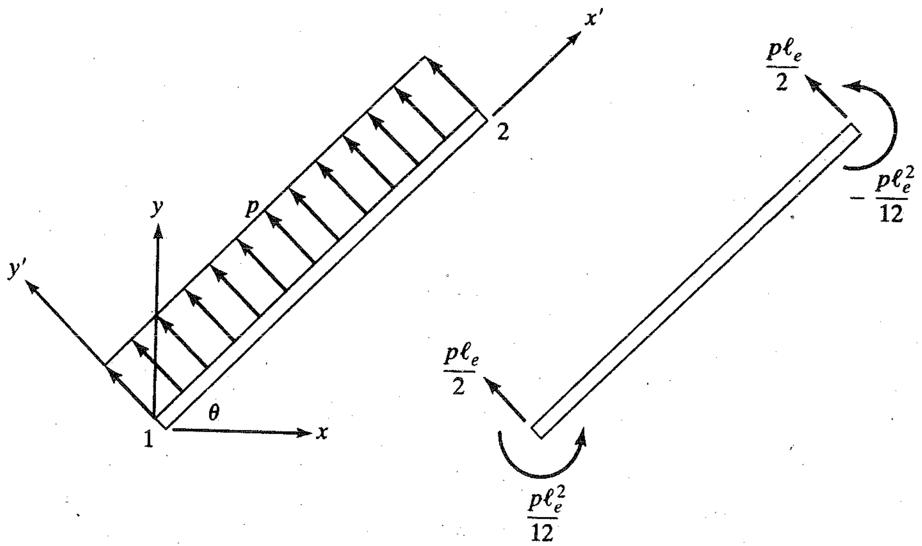

FIGURE 8.10 Distributed load on a frame element.

The values of **f** are added to the global load vector. Note here that positive p is in the y' direction.

The point loads and couples are simply added to the global load vector. On gathering stiffnesses and loads, we get the system of equations

$$\mathbf{KQ} = \mathbf{F}$$

where the boundary conditions are considered by applying the penalty terms in the energy or Galerkin formulations.

Example 8.2

Determine the displacements and rotations of the joints for the portal frame shown in Fig. E8.2.

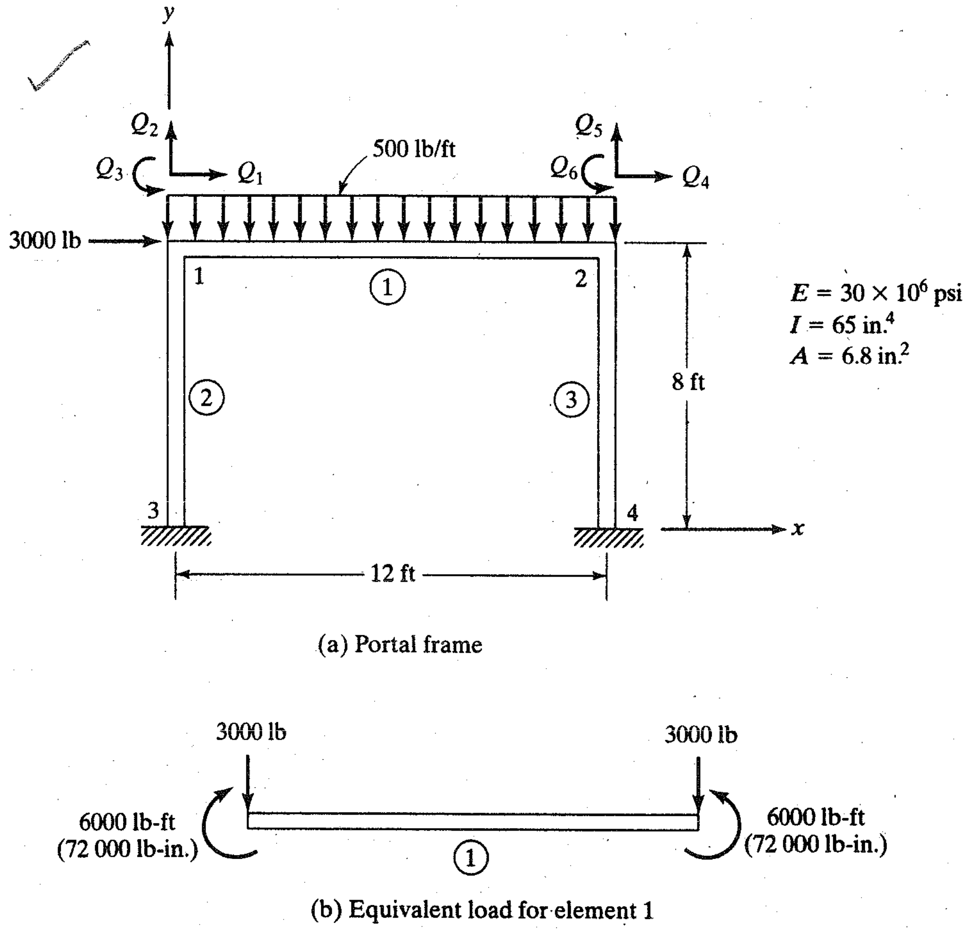

FIGURE E8.2 (a) Portal frame. (b) Equivalent load for Element 1.

Solution We follow the steps given below:

Step 1. Connectivity

The connectivity is as follows:

	Node	
Element No.	1	2
1	1	2
2	3	1
3	4	2

Step 2. Element Stiffnesses

Element 1. Using the matrix given in Eq. 8.45 and noting that $\mathbf{k}^1 = \mathbf{k}'^1$, we find that

$$\mathbf{k}^1 = 10^4 \times \begin{array}{c} \begin{array}{cccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{array} \\ \begin{bmatrix} 141.7 & 0 & 0 & -141.7 & 0 & 0 \\ 0 & 0.784 & 56.4 & 0 & -0.784 & 56.4 \\ 0 & 56.4 & 5417 & 0 & -56.4 & 2708 \\ -141.7 & 0 & 0 & 141.7 & 0 & 0 \\ 0 & -0.784 & -56.4 & 0 & 0.784 & -56.4 \\ 0 & 56.4 & 2708 & 0 & -56.4 & 5417 \end{bmatrix} \end{array}$$

Elements 2 and 3. Local element stiffnesses for elements 2 and 3 are obtained by substituting for E, A, I and ℓ_2 in matrix $\mathbf{k}'$ of Eq. 8.49:

$$\mathbf{k}'^2 = 10^4 \times \begin{bmatrix} 212.5 & 0 & 0 & -212.5 & 0 & 0 \\ 0 & 2.65 & 127 & 0 & -2.65 & 127 \\ 0 & 127 & 8125 & 0 & -127 & 4063 \\ -212.5 & 0 & 0 & 212.5 & 0 & 0 \\ 0 & -2.65 & -127 & 0 & 2.65 & -127 \\ 0 & 127 & 4063 & 0 & -127 & 8125 \end{bmatrix}$$

Transformation matrix **L.** We have noted that for element 1, $\mathbf{k} = \mathbf{k}$. For elements 2 and 3, which are oriented similarly with respect to the x- and y-axes, we have $\ell = 0, m = 1$. Then,

$$\mathbf{L} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Noting that $\mathbf{k}^2 = \mathbf{L}^{\mathrm{T}}\mathbf{k}'^2\mathbf{L}$, we get

$$\begin{array}{llll} e = 3 & Q_4 & Q_5 & Q_6 \\ e = 2 \rightarrow & Q_1 & Q_2 & Q_3 \end{array}$$

$$\mathbf{k} = 10^4 \times \begin{bmatrix} 2.65 & 0 & -127 & -2.65 & 0 & -127 \\ 0 & 212.5 & 0 & 0 & -212.5 & 0 \\ -127 & 0 & 8125 & 127 & 0 & 4063 \\ -2.65 & 0 & 127 & 2.65 & 0 & 127 \\ 0 & -212.5 & 0 & 0 & 212.5 & 0 \\ -127 & 0 & 4063 & 127 & 0 & 8125 \end{bmatrix}$$

Stiffness $\mathbf{k}^1$ has all its elements in the global locations. For elements 2 and 3, the shaded part of the stiffness matrix shown previously is added to the appropriate global locations of $\mathbf{K}$. The global stiffness matrix is given by

$$\mathbf{K} = 10^4 \times \begin{bmatrix} 144.3 & 0 & 127 & -141.7 & 0 & 0 \\ 0 & 213.3 & 56.4 & 0 & -0.784 & 56.4 \\ 127 & 56.4 & 13542 & 0 & -56.4 & 2708 \\ -141.7 & 0 & 0 & 144.3 & 0 & 127 \\ 0 & -0.784 & -56.4 & 0 & 213.3 & -56.4 \\ 0 & 56.4 & 2708 & 127 & -56.4 & 13542 \end{bmatrix}$$

From Fig. E8.2, the load vector can easily be written as

$$\mathbf{F} = \begin{Bmatrix} 3\,000 \\ -3\,000 \\ -72\,000 \\ 0 \\ -3\,000 \\ +72\,000 \end{Bmatrix}$$

The set of equations is given by

$$\mathbf{KQ} = \mathbf{F}$$

On solving, we get

$$\mathbf{Q} = \begin{Bmatrix} 0.092 \text{ in.} \\ -0.00104 \text{ in.} \\ -0.00139 \text{ rad} \\ 0.0901 \text{ in.} \\ -0.0018 \text{ in.} \\ -3.88 \times 10^{-5} \text{ rad} \end{Bmatrix}$$

■

8.8 THREE-DIMENSIONAL FRAMES

Three-dimensional frames, also called as *space frames*, are frequently encountered in the analysis of multistory buildings. They are also to be found in the modeling of car body and bicycle frames. A typical three-dimensional frame is shown in Fig. 8.11. Each node has six degrees of freedom (dofs) (as opposed to only three dofs in a plane frame). The dof numbering is shown in Fig. 8.11: for node J, dof $6J$–5, $6J$–4, and $6J$–3 represent